

Dynamics of Causal Fermion Systems

Field Equations and Correction Terms
for a New Unified Physical Theory

Dynamik kausaler Fermionensysteme

*Feldgleichungen und Korrekturterme
für eine neue vereinheitlichte Theorie*



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I dedicate this thesis to humanity in the hope that it will continue its bright path to establish a world which is just and good to all.

Abstract

The theory of causal fermion systems is a new physical theory which aims to describe a fundamental level of physical reality. Its mathematical core is the causal action principle. In this thesis, we develop a formalism which connects the causal action principle to a suitable notion of fields on space-time. We derive field equations from the causal action principle and find that the dynamics induced by the field equations conserve a symplectic form which gives rise to an Hamiltonian time evolution if the causal fermion system admits a notion of ‘time’. In this way, we establish the dynamics of causal fermion systems.

Remarkably, the causal action principle implies that there are correction terms to the field equations, which we subsequently derive and study. In particular, we prove that there is a stochastic and a non-linear correction term and investigate how they relate to the Hamiltonian time evolution. Furthermore, we give theorems which generalize the connection between symmetries and conservation laws in Noether’s theorems to the theory of causal fermion systems. The appearance of the particular correction terms is reminiscent of dynamical collapse models in quantum theory.

Kurzzusammenfassung

Die Theorie der kausalen Fermionensysteme ist eine kürzlich entwickelte physikalische Theorie, welche die fundamentalen Theorien in der Physik vereinheitlicht und somit einen neuen Vorschlag für die Beschreibung der grundlegenden Strukturen der physikalischen Realität darstellt. Die maßgebliche mathematische Struktur in dieser Theorie bildet das sogenannte kausale Wirkungsprinzip. Gegenstand dieser Arbeit ist es, einen Formalismus zu entwickeln, welcher aufbauend auf diesem abstrakten Prinzip eine Beschreibung liefert, die Parallelen zu den üblichen Theorien in der Physik hat: Die Dynamik von kausalen Fermionensystemen.

Ergebnis dieser Forschungen ist der sogenannte Jet-Formalismus von kausalen Fermionensystemen, welcher als verallgemeinerte physikalische Felder 1-Jets zu Grunde legt. Basierend auf den Euler-Lagrange Gleichungen des kausalen Wirkungsprinzips leiten wir Feldgleichungen ab und studieren deren Lösungen. Es zeigt sich, dass eine symplektische Form konstruiert werden kann, welche unter der von den Feldgleichungen induzierten Dynamik erhalten ist. Dies führt zur Definition einer Hamiltonschen Zeitentwicklung für kausale Fermionensysteme.

Bemerkenswerterweise führt das kausale Wirkungsprinzip zu Korrekturtermen für die Feldgleichungen, die wir im weiteren Verlauf der Arbeit untersuchen. Unsere Theoreme etablieren einen stochastischen und einen nicht-linearen Korrekturterm, sowie deren Beziehung zu markoskopischen Gleichungen und zur eingangs erwähnten symplektischen Form. Das Auftreten dieser speziellen Korrekturterme deutet auf Parallelen zu dynamischen Kollaps-Theorien in der nicht-relativistischen Quantenmechanik hin. Die Arbeit enthält außerdem Theoreme, welche den durch die Noetherschen Theoreme gegebenen Zusammenhang zwischen Symmetrien und Erhaltungsgrößen auf die Theorie der kausalen Fermionensysteme übertragen.

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Chapter 1

Introduction

The theory of causal fermion systems is a new physical theory which aims to describe a fundamental level of physical reality. Since it gives quantum mechanics, general relativity and quantum field theory as limiting cases ([Fin16b, Fin14]), it constitutes a unification of the currently accepted fundamental physical theories.

The mathematical structure of the theory of causal fermion systems differs substantially from the mathematical structure of quantum mechanics, quantum field theory and general relativity. The basic object is a *causal fermion system*, defined as a triple $(\mathcal{H}, \mathcal{F}, \rho)$, where \mathcal{H} is a separable complex Hilbert space, $\mathcal{F} \subset L(\mathcal{H})$ is the set of all those self-adjoint linear operators on \mathcal{H} which (counting multiplicities) have at most n positive and at most n negative eigenvalues, where n is a parameter of the theory, and ρ is a Borel measure on \mathcal{F} . Its basic principle is the *causal action principle*, which consists of minimizing the action

$$\mathcal{S}(\rho) = \int_{\mathcal{F}} \int_{\mathcal{F}} \mathcal{L}(x, y) d\rho(x) d\rho(y) \quad (1.1)$$

under variations of the measure ρ , taking into account additional constraints, for example $\rho(\mathcal{F})$ to be constant. Here, $\mathcal{L}(x, y) := \frac{1}{4n} \sum_{i,j=1}^{2n} (|\lambda_i^{xy}| - |\lambda_j^{xy}|)^2$ is the Lagrangian of the theory, where λ_i^{xy} denote the eigenvalues of the operator product xy for $x, y \in \mathcal{F}$.

The goal of this thesis is to connect this mathematical structure to a description which is more similar to the currently used physical theories: A formulation in terms of fields on space-time, where in the context of causal fermion systems, for physical reasons, space-time M is defined as

$$M := \text{supp } \rho.$$

To achieve this goal of constructing an effective description of the theory in terms of fields on space-time, we consider variations of a measure ρ which are infinitesimally described by a function $b : \mathcal{F} \rightarrow \mathbb{R}$ and by a vector field v on \mathcal{F} . Combined, these two form a *jet* $\mathfrak{v} := (b, v)$. It turns out that the restriction of jets \mathfrak{v} to $M \subset \mathcal{F}$ yields a fruitful and satisfying notion of fields on space-time in this context.

Based on jets, we develop the so-called *jet-formalism* of causal fermion systems. Starting from the Euler-Lagrange equations of the causal action principle, we derive field equations for jets on space-time and show that those field-equations give rise to a symplectic form. This symplectic form is expressed in terms of so-called *surface layer integrals* which

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generalize surface integrals to causal fermion systems. If space-time M contains a suitable notion of Cauchy surfaces (and hence time), the surface layer integrals can be associated to Cauchy surfaces at different times. We prove that this yields a symplectic form which is conserved with respect to the time-evolution induced by the field equations. We call this a *Hamiltonian time evolution*.

Importantly, the causal action principle implies that there are correction terms to the field equations, which we subsequently derive and study. In particular, we prove that there is a stochastic and a non-linear correction term and investigate how they relate to the Hamiltonian time evolution. In this way, we establish the *dynamics of causal fermion systems*. An important first step in our investigations is the construction of theorems which generalize Noether's theorems to the setting of causal fermion systems.

Throughout this thesis, we refer to two different settings. The first is the *setting of causal fermion systems* which we have described above (second paragraph of the introduction). A second setting is the *general setting*, where we assume that \mathcal{F} is a smooth, finite dimensional manifold, ρ is a Borel measure on \mathcal{F} and $\mathcal{L} : \mathcal{F} \times \mathcal{F} \rightarrow \mathbb{R}_0^+$ is a general function. The name of this setting comes about because it is more general than the setting of causal fermion systems (and also easier accessible). Nevertheless, the physically interesting case is the setting of causal fermion systems.

When working in the general setting, we use different regularity assumptions for the Lagrangian \mathcal{L} , depending on what is necessary to make all constructions well-defined. To make this easily apparent, we introduce the names

- *lower semi-continuous setting*, if we work in the general setting and \mathcal{L} is assumed to be lower semi-continuous,
- *Lipschitz-continuous setting*, if we work in the general setting and \mathcal{L} is assumed to be Lipschitz-continuous, and
- *smooth setting*, if we work in the general setting and \mathcal{L} is assumed to be smooth.

Furthermore, we define the

- *compact setting* as the Lipschitz-continuous setting with the additional assumption of \mathcal{F} being compact.

For simplicity and clarity, in this introduction, we mostly outline our results and methods in the general setting. As usual, we aim for a concise presentation and hence omit technical details, referring to later parts of this thesis whenever necessary.

This dissertation and its introduction are structured as follows. In a first step towards finding a connection to fields on space-time, we prove generalizations of Noether's theorems to causal fermion systems. An outline of this work is given in Section 1.1, all details can be found in Chapter 3. Chapter 4 contains the jet-formalism of causal fermion systems, the derivation of field equations and the construction of the above-mentioned symplectic form and of the Hamiltonian time evolution. The results are outlined in Section 1.2. Chapter 5 is concerned with the derivation of correction terms to the field equations as well as the investigation of its implications. This is outlined in Section 1.3. We conclude this introduction in Section 1.4 by explaining the connection of the results of this thesis to foundations of quantum theory. This connection has been one of the main driving forces behind these investigations.

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Chapter 2 is devoted to give an easily accessible introduction to the theory of causal fermion systems, focussing on the basic concepts and the general physical picture behind. The full mathematical setup of the setting of causal fermion systems is introduced in Section 3.3.1.

We remark that Chapter 2 has been published as [FK15], Chapter 3 as [FK16] and Chapter 4 as [FK17]. Even though these chapters relate to one another, they are self-contained, allowing the reader to leap forward to any one of them at any point. Being the most recent development, Chapter 5 has not yet been published. It is not self-contained, but a reading of Sections 4.1.1, 4.1.2 as well as 4.3.1 to 4.3.3 contains all necessary prerequisites.

1.1 Noether-Like Theorems

In modern physics, the connection between symmetries and conservation laws is of central importance. For continuous symmetries, this connection is made mathematically precise by Noether's theorem [Noe18].

In a first part of this thesis, we explore symmetries and the resulting conservation laws in the framework of causal fermion systems. The difficulty is that a priori, it is not clear at all how the definitions of symmetries in contemporary physics (cf. Section 3.1.1) can be generalized to causal fermion systems, nor which form conservation laws could take. We now present the answers to these questions in the compact setting.

To formalize the notion of symmetries, we consider mappings (3.14),

$$\Phi : (-\tau_{\max}, \tau_{\max}) \times M \rightarrow \mathcal{F} \quad \text{with} \quad \Phi(0, \cdot) = \mathbb{1} ,$$

denoted as $\Phi_\tau(x)$, which are called *variations* of M in \mathcal{F} .

It turns out that three definitions of symmetries can be considered. A variation Φ is a *symmetry of the Lagrangian* (Definition 3.2.2) if

$$\mathcal{L}(x, \Phi_\tau(y)) = \mathcal{L}(\Phi_{-\tau}(x), y) \quad \text{for all } \tau \in (-\tau_{\max}, \tau_{\max}) \text{ and } x, y \in M .$$

It is a *symmetry of the universal measure* (Definition 3.2.4) if

$$(\Phi_\tau)_* \rho = \rho \quad \text{for all } \tau \in (-\tau_{\max}, \tau_{\max}) ,$$

where $(\Phi_\tau)_* \rho$ is the push-forward measure, and a *generalized integrated symmetry* (Definition 3.2.7) if

$$\int_M d\rho(x) \int_\Omega d\rho(y) \left(\mathcal{L}(\Phi_\tau(x), y) - \mathcal{L}(x, y) \right) = 0 .$$

Generalized integrated symmetries unite symmetries of the Lagrangian and symmetries of the universal measure, cf. Section 3.2.3. All of these notions of symmetries give rise to conservation laws (Theorems 3.2.3, 3.2.5, 3.2.8).

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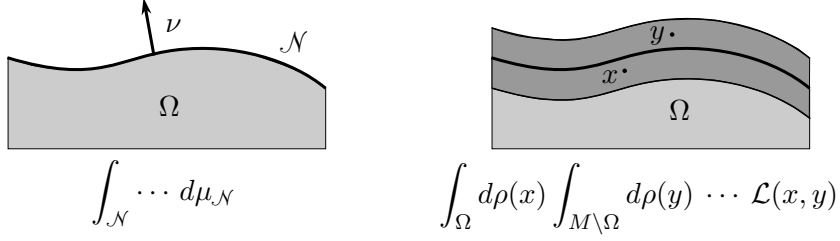


Figure 1.1: A surface integral and a corresponding surface layer integral.

Theorem. *Let ρ be a measure which minimizes the causal variational principle and Φ_τ a symmetry of the Lagrangian which is differentiable as in Definition 3.2.1. Then for any compact subset $\Omega \subset M$, we have*

$$\frac{d}{d\tau} \int_\Omega d\rho(x) \int_{M \setminus \Omega} d\rho(y) \left(\mathcal{L}(\Phi_\tau(x), y) - \mathcal{L}(\Phi_{-\tau}(x), y) \right) \Big|_{\tau=0} = 0. \quad (1.2)$$

Theorem. *Let ρ be a minimizing measure and Φ_τ a symmetry of the universal measure which is differentiable as specified in Definition 3.2.1. For any compact subset $\Omega \subset M$,*

$$\frac{d}{d\tau} \int_\Omega d\rho(x) \int_{M \setminus \Omega} d\rho(y) \left(\mathcal{L}(\Phi_\tau(x), y) - \mathcal{L}(x, \Phi_\tau(y)) \right) \Big|_{\tau=0} = 0. \quad (1.3)$$

Theorem. *Let ρ be a minimizing measure and Φ_τ a generalized integrated symmetry which is differentiable as specified in Definition 3.2.1. Then for any compact subset $\Omega \subset M$, (1.3) holds.*

All of these conservation laws are examples of *surface layer integrals*, as introduced in Section 3.1.3. The structure of surface layer integrals can be understood most easily in the special situation that the Lagrangian is of short range in the sense that $\mathcal{L}(x, y)$ vanishes unless x and y are close together. In this situation, we only get a contribution to the double integrals (1.2) and (1.3) if both x and y are close to the boundary $\partial\Omega$ of Ω , as indicated by the dark grey region in Figure 1.1. Therefore, surface layer integrals can be understood as an adaptation of surface integrals to the setting of causal variational principles (for a more detailed explanation see Section 3.1.3). Figure 1.1 illustrates the similarity between surface integrals (left) and surface layer integrals (right).

In Section 3.3, we give the definitions of the above symmetries in the setting of causal fermion systems and derive the corresponding conservation laws. For brevity we do not repeat this here. Building on those generalizations, in Section 3.4, we show that a transformation

$$\Phi_\tau : \mathbb{R} \times \mathcal{F} \rightarrow \mathcal{F}, \quad \Phi(\tau, x) = \mathcal{U}_\tau x \mathcal{U}_\tau^{-1}.$$

with

$$\mathcal{U}_\tau := \exp(i\tau \mathcal{A}),$$

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where \mathcal{A} is a bounded symmetric operator on \mathcal{H} , yields a symmetry of the Lagrangian (cf. Lemma 3.4.1). We prove that in a limit of the theory of causal fermion systems, evaluation of the conservation law corresponding to (1.2) for this symmetry yields current conservation for the Dirac equation (Theorem 3.4.4).

Theorem. *Let $(\mathcal{H}, \mathcal{F}, \rho^\varepsilon)$ be local minimizers of the causal action which describe the Minkowski vacuum (3.60). Considering the limiting procedure explained in Figure 3.2 and taking the continuum limit, the conservation laws of Theorem 3.4.2 go over to a linear combination of the probability integrals in every generation.*

Finally, in Section 3.5, we establish a connection to energy-momentum conservation. To this end, we give a suitable definition of Killing symmetries (Definition 3.5.1) and derive a corresponding conservation law (Theorem 3.5.2). Theorem 3.5.3 establishes that the conservation law associated to Killing symmetries of causal fermion systems indeed yields energy-momentum conservation. Thus the conservation laws of charge and energy-momentum can be viewed as special cases of more general conservation laws which are intrinsic to causal fermion systems.

We remark that our conservation laws also apply to “quantum space-times” ([FG12]) which cannot be approximated by a Lorentzian manifold.

1.2 Hamiltonian Formulation and Linearized Field Equations

In Chapter 4, we develop a formalism to define the dynamics of the theory of causal fermion systems, where, as mentioned above, the term ‘dynamics’ loosely refers to objects which propagate on space-time. The difficulty here is that the theory of causal fermion systems, as introduced above, has a structure which is completely distinct from any formulation in terms of dynamics. Hence, even which objects to consider is a priori unclear, let alone how to construct field equations and conserved quantities. We now explain our constructions and results in the general setting.

Let ρ be a minimizer of the causal variational principle (for mathematical details see Section 4.1.1 below). The key step towards describing the causal variational principle in terms of a Hamiltonian time evolution is to consider variations of ρ described by a diffeomorphism $F : \mathcal{F} \rightarrow \mathcal{F}$ and a weight function $f : \mathcal{F} \rightarrow \mathbb{R}_0^+$. More precisely, we consider families $(F_\tau)_{\tau \in \mathbb{R}}$ and $(f_\tau)_{\tau \in \mathbb{R}}$ of such diffeomorphisms and weight functions and form a corresponding family $(\rho_\tau)_{\tau \in \mathbb{R}}$ of measures by

$$\rho_\tau = (F_\tau)_*(f_\tau \rho). \quad (1.4)$$

Here $(F_\tau)_*\mu$ denotes the push-forward of the measure μ . Thus the measure ρ_τ is obtained from ρ by first multiplying with the weight function f_τ and then “transporting” the resulting measure with the diffeomorphism F_τ on \mathcal{F} .

Infinitesimal versions of the variation (1.4) consist of a scalar part corresponding to the τ -derivative of f_τ and a vectorial part corresponding to the τ -derivative of F_τ . Thus

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variations of the form (1.4) can be described infinitesimally by a pair (b, v) of a real-valued function and a vector field. It turns out that in the context of causal fermion systems, the pairs (b, v) , if restricted to M , are suitable candidates for physical fields on space-time. Put differently, pairs (b, v) can be viewed as generalized physical fields. We remark at this point that the definition of *space-time* M as $M := \text{supp } \rho \subset \mathcal{F}$ indeed generalizes the usual notion of space-time (being Minkowski space or a Lorentzian manifold) in the setting of causal fermion systems (for details see [Fin16b, Section 1.2] or [FG12, Sections 4 and 5]).

In order to have a short name which cannot be confused with common notions in physics, we refer to the pairs (b, v) as *jets*, being elements of the corresponding *jet space*¹

$$\mathfrak{J} := \{ \mathfrak{v} = (b, v) \text{ with } b : \mathcal{F} \rightarrow \mathbb{R} \text{ and } v \in \Gamma(\mathcal{F}) \}$$

(see (4.26) and (4.54)).

We are interested in variations of the form (1.4) which are minimizers of the causal action also for $\tau \neq 0$. Such “families of minimizers” are of interest because in the theory of causal fermion systems, they correspond to variations which satisfy the physical equations. The requirement of ρ_τ being a minimizer for all τ gives rise to conditions for the jet $\mathfrak{v} = \partial_\tau \rho_\tau|_{\tau=0} \in \mathfrak{J}$ describing the infinitesimal variation. In view of the similarities and the correspondence to classical field theory (as worked out in [Fin16b, §1.4.1 and Chapters 3-5] and [Fin17]), we refer to these conditions as the *linearized field equations*. In order to specify the linearized field equations, we introduce the function

$$\ell(x) = \int_{\mathcal{F}} \mathcal{L}(x, y) d\rho(y) - \frac{\nu}{2} \quad (1.5)$$

which is used in the Euler-Lagrange equations of causal variational principles (Section 4.1.2). The parameter $\nu \geq 0$ is the Lagrange multiplier corresponding to the volume constraint. We define the differentiable one-jets $\mathfrak{J}^{\text{diff}} \subset \mathfrak{J}$ to consists of all those jets $\mathfrak{u} = (a, u)$ for which ℓ is differentiable in the direction of u on M ,

$$\mathfrak{J}^{\text{diff}} := \{ \mathfrak{u} = (a, u) \in \mathfrak{J} \mid D_u^+ \ell|_M = D_u^- \ell|_M \},$$

where D_u^+ denotes the right directional semi-derivative (4.48) and D_u^- denotes the left directional semi-derivative (4.72). Also, we define the *derivative* $\nabla_{\mathfrak{v}}$ *in the direction of a one-jet* $\mathfrak{v} = (b, v)$ as as a combination of multiplication and differentiation,

$$\nabla_{\mathfrak{v}} \eta(x) = b(x) \eta(x) + D_v \eta(x)$$

(where D_v is the usual directional derivative of functions on \mathcal{F}), and similarly for semi-derivatives $\nabla_{\mathfrak{v}}^+$ and $\nabla_{\mathfrak{v}}^-$ (cf. (4.52)). The notation $\nabla_{1, \mathfrak{v}}$ and $\nabla_{2, \mathfrak{v}}$ denotes partial derivatives acting on the first and second argument of $\mathcal{L}(x, y)$, respectively.

¹The connection to jets in differential geometry (see for example [Sau89]) is obtained by considering real-valued functions on \mathcal{F} . Then their one-jets are elements in $C^\infty(\mathcal{F}) \oplus \Gamma(\mathcal{F}, T^*\mathcal{F})$. Identifying the cotangent space with the tangent space gives our jet space \mathfrak{J} .

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For differentiable one-jets, the Euler-Lagrange equations of the causal variational principle imply the so-called *weak Euler-Lagrange equations*,

$$\nabla_{\mathbf{u}} \ell(x) = 0 \quad \text{for all } x \in M \text{ and } \mathbf{u} \in \mathfrak{J}^{\text{diff}} \quad (1.6)$$

(cf. 4.55), which are the starting point of the derivation of the linearized field equations.

Using the above definitions, the *linearized field equations* can be written as

$$\nabla_{\mathbf{u}} \left(\int_M (\nabla_{1,\mathbf{v}} + \nabla_{2,\mathbf{v}}) \mathcal{L}(x, y) d\rho(y) - \nabla_{\mathbf{v}} \frac{\nu}{2} \right) = 0 \quad (1.7)$$

for all $\mathbf{u} \in \mathfrak{J}^{\text{test}}$ and $x \in M$. (For details see Lemma 4.2.1 and (4.68).)

Here, the *test jets* $\mathfrak{J}^{\text{test}} \subset \mathfrak{J}^{\text{diff}}$ are defined as a subspace of the differentiable one-jets used to test the requirement of minimality in a weak sense. This so-called *weak evaluation of the Euler-Lagrange equations* is an important mathematical and physical concept because by choosing $\mathfrak{J}^{\text{test}}$ appropriately, one can restrict attention to the part of the information contained in the Euler-Lagrange equations which is relevant for the application in mind. In order to illustrate how this works, we give a typical example: For the description of macroscopic physics, one would like to disregard effects which come into play only on the Planck scale. To this end, one chooses $\mathfrak{J}^{\text{test}}$ as a space of jets which vary only on the macroscopic scale, so that “fluctuations on the Planck scale are filtered out”.

The above results hold in the general setting. Next, we restrict to the smooth setting, i.e. we assume that \mathcal{L} is smooth, cf. Section 4.2. The assumptions in this section ensure that $\mathfrak{J}^{\text{diff}} = \mathfrak{J}$. We consider the set \mathcal{B} of all measures of the form (1.4) which satisfy the weak Euler-Lagrange equations (1.6) and assume that this set is a smooth Fréchet manifold. Then infinitesimal variations of (1.4) which are solutions of (1.7) are vectors of the tangent space $T_{\rho} \mathcal{B}$. In this setting, we show that the structure of the causal variational principle gives rise to a *symplectic form* on \mathcal{B} (Section 4.2.3). Namely, for any $\mathbf{u}, \mathbf{v} \in T_{\rho} \mathcal{B}$ and $x, y \in M$, let

$$\sigma_{\mathbf{u}, \mathbf{v}}(x, y) := \nabla_{1,\mathbf{u}} \nabla_{2,\mathbf{v}} \mathcal{L}(x, y) - \nabla_{1,\mathbf{v}} \nabla_{2,\mathbf{u}} \mathcal{L}(x, y) .$$

Given a compact subset $\Omega \subset \mathcal{F}$, we define the bilinear form

$$\sigma_{\Omega} : T_{\rho} \mathcal{B} \times T_{\rho} \mathcal{B} \rightarrow \mathbb{R}, \quad \sigma_{\Omega}(\mathbf{u}, \mathbf{v}) = \int_{\Omega} d\rho(x) \int_{M \setminus \Omega} d\rho(y) \sigma_{\mathbf{u}, \mathbf{v}}(x, y) . \quad (1.8)$$

Thus here we again make use of the structure of a surface layer integral. The following theorem holds (Theorem 4.2.3).

Theorem. *For any compact subset $\Omega \subset \mathcal{F}$, the surface layer integral (1.8) vanishes for all $\mathbf{u}, \mathbf{v} \in T_{\rho} \mathcal{B}$.*

This theorem has the following connection to conservation laws. Let us assume that M admits a sensible notion of “spatial hypersurfaces” and that the jets $\mathbf{u}, \mathbf{v} \in T_{\rho} \mathcal{B}$ have

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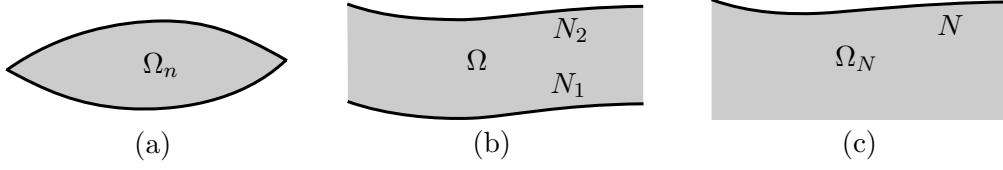


Figure 1.2: Choices of space-time regions.

suitable decay properties at spatial infinity. Then, given two such hypersurfaces N_1 and N_2 , one can choose a sequence $\Omega_n \subset M$ of compact sets which form an exhaustion of the space-time strip between N_1 and N_2 (see Figure 1.2 (a) and (b)). Let us denote this space-time strip by Ω . Considering the surface layer integrals (1.8) for Ω_n and passing to the limit, the above theorem implies that also the surface layer integral corresponding to Ω vanishes, so that the contributions from N_1 and N_2 coincide. (This is made precise in Section 4.2.3.) Let us define a surface layer integral corresponding to any spatial hypersurface N as

$$\sigma_N(\mathbf{u}, \mathbf{v}) := \int_{\Omega_N} d\rho(x) \int_{M \setminus \Omega_N} d\rho(y) \sigma_{\mathbf{u}, \mathbf{v}}(x, y) \quad (1.9)$$

where Ω_N is a set with $\partial\Omega_N = N$ as shown in Figure 1.2 (c). Then the above theorem implies that this quantity is well-defined and that $\sigma_N(\mathbf{u}, \mathbf{v}) = \sigma_{N'}(\mathbf{u}, \mathbf{v})$ for any two such hypersurfaces N and N' . In other words, this surface layer integral is independent of the particular choice of hypersurface N .

If the hypersurfaces N are similar to a foliation of Cauchy surfaces N_t and if the parameter t of the foliation can be interpreted as time, the last theorem implies that the bilinear form σ_{N_t} is preserved under the time evolution. This is what we call *Hamiltonian time evolution*.

To avoid misunderstandings, we point out that, in contrast to classical field theory, in our setting the time evolution is *not defined infinitesimally* by a Hamiltonian or a Hamiltonian vector field (this is obvious from the fact that causal fermion systems allow for the description of discrete space-times, where a continuous time evolution makes no sense). Instead, the time evolution should be thought of as a mapping from the jets in a surface layer around N_1 to the jets in a surface layer around N_2 . This mapping is a *symplectomorphism* with regard to σ_{N_1} and σ_{N_2} , respectively. For clarity, we also note that it is essential here that \mathcal{F} and M are non-compact because otherwise, Theorem 4.2.3 would immediately imply that $\sigma_N(\mathbf{u}, \mathbf{v}) \equiv 0$. For non-compact M , however, Theorem 4.2.3 only applies to the difference $\sigma_{N_1} - \sigma_{N_2}$, thereby giving a conservation law for a non-trivial surface layer integral.

The independence of (1.9) from N allows us to define a bilinear form σ on \mathcal{B} by

$$\sigma : T_\rho \mathcal{B} \times T_\rho \mathcal{B} \rightarrow \mathbb{R}, \quad \sigma(\mathbf{u}, \mathbf{v}) := \sigma_N(\mathbf{u}, \mathbf{v})$$

for an arbitrary choice of hypersurface N . This bilinear form turns out to be closed (see

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Lemma 4.2.4), thus defining a *presymplectic form* on \mathcal{B} . Finally, by restricting σ to a suitable subspace of $T_\rho\mathcal{B}$, we arrange that σ is non-degenerate, giving a *symplectic form* (see end of Section 4.2.3).

In the theory of causal fermion systems, the Lagrangian is not a smooth, but merely a Lipschitz-continuous function (see Section 3.3.1, [Fin16b, Section 1.1] and [FS13]). In order to cover this situation, in Section 4.3 we treat the more general *lower semi-continuous setting* where we have a lower semi-continuous Lagrangian. Our main motivation for this generalization compared to Lipschitz-continuity is that many simple examples are easier to state if we allow for discontinuities of the Lagrangian. After defining jet spaces as infinitesimal versions of families of solutions similar as described above (see Sections 4.3.1 and 4.3.2) and establishing the necessary conditions for the linearized field equations to be well-defined (see Definition 4.3.3), we again establish a conservation law for the bilinear form $\sigma_\Omega(\mathbf{u}, \mathbf{v})$ (see Theorem 4.3.5).

Our results apply directly to the setting of causal fermion systems if one restricts the set of operators \mathcal{F} to operators of maximal rank, i.e. with n positive and n negative eigenvalues. This is explained in detail in Section 4.1.4.

In Section 4.4, we illustrate our constructions in an example which is simple enough for an explicit analysis but nevertheless captures some features of a physical field theory. We choose $\mathcal{L}(x, y)$ in such a way that the minimizing measure is supported on a two-dimensional lattice. This reflects a general feature of the theory of causal fermion systems that space-time “discretizes itself” on the Planck scale, thus avoiding the ultraviolet divergences of quantum field theory (see [FGS12, Section 4]).

1.3 Stochastic and Non-Linear Correction Terms

Chapter 5 is devoted to finding correction terms to the linearized field equations. The *motivation* to look for correction terms in the theory of causal fermion systems is the following. The linearized field equations (1.7) are a consequence of the weak Euler-Lagrange equations (1.6). But since the Lagrangian \mathcal{L} in the theory of causal fermion systems (and hence the function ℓ as defined in (1.5)) are not differentiable but only Lipschitz-continuous, instead of (1.6), we only have

$$\nabla_{\mathbf{u}}^+ \ell(x) \geq 0 \quad \text{for all } x \in M \text{ and } \mathbf{u} \in \mathfrak{J}|_M, \quad (1.10)$$

where the semi-derivative in the direction of a jet $\mathbf{u} = (a, u) \in \mathfrak{J}$ is defined as (cf. (4.51))

$$\nabla_{\mathbf{u}}^+ \ell(x) := a(x) \ell(x) + (D_u^+ \ell)(x). \quad (1.11)$$

Due to the inequality in (1.10), one can expect additional terms to appear in (1.7).

Whereas this motivation to look for correction terms is simple, it turns out that many ways to derive field *equations* which include a correction term turn out unsuccessful. The successful way turns out to be deeply rooted in the structure of the theory of causal fermion systems. Put in very simple terms, one uses the definition of ℓ as in (1.5),

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evaluated for a family of measures as (1.4),

$$\ell_\tau(x) := \int_{\mathcal{F}} \mathcal{L}(x, y) d\rho_\tau(y) - \frac{\nu}{2}.$$

Composing this definition with the flow Φ of the vectorial component w of jet $\mathfrak{w} \in \mathfrak{J}$ in a suitable way, and taking semi-derivatives with respect to the parameters of the family (1.4) and of the flow Φ gives the *non-differentiable linearized field equations* (Proposition 5.2.1),

$$\frac{1}{2} (\nabla_{\mathfrak{w}}^+ - \nabla_{-\mathfrak{w}}^+) \int_M d\rho(y) (\nabla_{1,\mathfrak{v}}^+ + \nabla_{2,\mathfrak{v}}^+) \mathcal{L}(x, y) - \nabla_{\mathfrak{w}}^+ \nabla_{\mathfrak{v}}^+ \frac{\nu}{2} = \chi_{\mathfrak{w},\mathfrak{v}}(x) \quad (1.12)$$

for all $\mathfrak{w} \in \mathfrak{J}$ and $x \in M$, where the term $\chi_{\mathfrak{w},\mathfrak{v}}(x)$ is defined as

$$\chi_{\mathfrak{w},\mathfrak{v}}(x) = \frac{1}{2} \frac{d^+}{ds|_0} \frac{d^+}{d\tau|_0} f_\tau(x) \left(\ell_\tau(F_\tau(\Phi_s(x))) - \ell_\tau(F_\tau(\Phi_{-s}(x))) \right), \quad (1.13)$$

and where the subscripts indicate that the semi-derivatives are evaluated at $s = \tau = 0$. Thus $\chi_{\mathfrak{w},\mathfrak{v}}(x)$ is the desired correction term!

The explanations in Section 5.2 and Example 5.2.2 show that $\chi_{\mathfrak{w},\mathfrak{v}}(x)$ is a term which has a varying sign as x changes. Since this sign depends on the microscopic structure of space-time M and the point-wise behaviour of ℓ , $\chi_{\mathfrak{w},\mathfrak{v}}(x)$ can be interpreted as a *stochastic term*. Note that if $\int_M d\rho(y) (\nabla_{1,\mathfrak{v}}^+ + \nabla_{2,\mathfrak{v}}^+) \mathcal{L}(x, y)$ exists and is differentiable, the left hand side of (1.12) is equal to the left hand side of the linearized field equations (1.7). Hence it indeed constitutes a generalization thereof.

The appearance of a correction term brings up the question of why a corresponding correction term is not apparent in contemporary experiments. We address this question in Section 5.3 by defining the following physically motivated assumption (Definition 5.3.2).

Definition. ‘Symmetric derivatives vanish macroscopically’ if for every minimizer ρ of the causal variational principle, every vector field $w \in \Gamma(T\mathcal{F})$ and every macroscopic region $\tilde{\Omega} \in \mathcal{M}$,

$$\frac{1}{2} \int_{\tilde{\Omega}} (D_w^+ - D_{-w}^+) \ell(x) d\rho(x) = 0. \quad (1.14)$$

Here, we assume that the specification of which subsets of M are macroscopic is part of the data of any application of the theory. (But cf. Remark 5.5.4.) Formally, we assume that this data is given as a set \mathcal{M} of subsets of \mathcal{F} , cf. Definition 5.3.1. The assumption of symmetric derivatives to vanish macroscopically is reasonable because the integrand has a varying sign, and hence the different contributions at different space-time points, if summed over a macroscopic region, might cancel.

Proposition 5.3.5 shows that if symmetric derivatives vanish macroscopically, the

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stochastic term vanishes macroscopically as well,

$$\int_{\tilde{\Omega}} \chi_{\mathfrak{w}, \mathfrak{v}}(x) d\rho(x) = 0. \quad (1.15)$$

(The same result would hold if the right hand side of (1.14) and (1.15) were replaced by ε .) Hence we can justify the above assumption a posteriori as an explanation of why correction terms to field equations are currently not observed. We point out, however, that the assumption could very well be wrong – investigations on minimizers of the causal action principle could prove that (1.14) does not vanish, cf. Chapter 6. In this way, the theory of causal fermion systems could allow to deduce what counts as a ‘macroscopic region’ from first principles.

After studying the connection to the differentiable case of Chapter 4 in detail in Section 5.4, we investigate the symplectic form (1.9) in the present setting. In contrast to the theorem mentioned in Section 1.2, it does not vanish, but additional terms appear, as shown by Theorem 5.5.1.

Theorem. *Let \mathfrak{w} and \mathfrak{v} be solutions of the non-differentiable linearized field equations (1.12). Then for any compact $\Omega \in \Sigma(\mathcal{F})$, the symplectic form (1.8) satisfies*

$$\sigma_{\Omega}(\mathfrak{w}, \mathfrak{v}) = \int_{\Omega} \tilde{\chi}_{\mathfrak{w}, \mathfrak{v}}(x) d\rho(x) - \int_{\Omega} \tilde{\nabla}_{[\mathfrak{w}, \mathfrak{v}]} \ell(x) d\rho(x),$$

where

$$\tilde{\chi}_{\mathfrak{w}, \mathfrak{v}}(x) = \frac{1}{2} (\chi_{\mathfrak{w}, \mathfrak{v}}(x) - \chi_{\mathfrak{w}, -\mathfrak{v}}(x) - \chi_{\mathfrak{v}, \mathfrak{w}}(x) + \chi_{\mathfrak{v}, -\mathfrak{w}}(x))$$

and

$$\tilde{\nabla}_{[\mathfrak{w}, \mathfrak{v}]} = \frac{1}{2} (\nabla_{[\mathfrak{w}, \mathfrak{v}]}^+ - \nabla_{-[\mathfrak{w}, \mathfrak{v}]}^+).$$

This theorem says that the Hamiltonian time evolution is broken on the microscopic level by the stochastic term and by the non-differentiability of ℓ . However, if symmetric derivatives vanish macroscopically, the Hamiltonian time evolution is conserved macroscopically, as shown by Proposition 5.5.2 (cf. Remark 5.5.4).

Theorem. *If symmetric derivatives vanish macroscopically, for any solutions \mathfrak{w} and \mathfrak{v} of the non-differentiable linearized field equations and any compact macroscopic region $\tilde{\Omega}$,*

$$\sigma_{\tilde{\Omega}}(\mathfrak{w}, \mathfrak{v}) = 0.$$

Finally, in Section 5.6, we derive quadratic corrections to the linearized field equations. Using the notations

$$\nabla_{\mathfrak{v}, \mathfrak{v}}^+ := \ddot{f}_0 + 2\dot{f}_0 D_v^+ + D_v^+ D_v^+$$

for any family (1.4) with generator $\mathfrak{v} = (\dot{f}_0, v) \in \mathfrak{J}$ (where the dot indicates a τ -derivative),

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as well as

$$\tilde{\nabla}_{\mathfrak{w}} := \frac{1}{2}(\nabla_{\mathfrak{w}}^+ - \nabla_{-\mathfrak{w}}^+),$$

we can thus specify the main theorem of Chapter 5:

Theorem. (Full non-differentiable field equations to second order)

For every family (1.4) of minimizers with generator \mathfrak{v} , any $x \in M$ and any $\mathfrak{w} \in \mathfrak{J}$, we have

$$\begin{aligned} & \tilde{\nabla}_{\mathfrak{w}} \int_M d\rho(y) (\nabla_{1,\mathfrak{v}}^+ + \nabla_{2,\mathfrak{v}}^+) \mathcal{L}(x, y) - \tilde{\nabla}_{\mathfrak{w}} \nabla_{\mathfrak{v}}^+ \frac{\nu}{2} \\ & + \tilde{\nabla}_{\mathfrak{w}} \int_M d\rho(y) \left(\nabla_{1,\mathfrak{v},\mathfrak{v}}^+ + \nabla_{2,\mathfrak{v},\mathfrak{v}}^+ + 2 \nabla_{1,\mathfrak{v}}^+ \nabla_{2,\mathfrak{v}}^+ \right) \mathcal{L}(x, y) - \tilde{\nabla}_{\mathfrak{w}} \nabla_{\mathfrak{v},\mathfrak{v}}^+ \frac{\nu}{2} \quad (1.16) \\ & = \chi_{\mathfrak{w},\mathfrak{v}}(x) + \chi_{\mathfrak{w},\mathfrak{v}}^{(2)}(x), \end{aligned}$$

where $\chi_{\mathfrak{w},\mathfrak{v}}(x)$ is as in (1.13) and

$$\chi_{\mathfrak{w},\mathfrak{v}}^{(2)}(x) = \frac{1}{2} \frac{d^+}{ds|_0} \frac{d^2}{d\tau^2|_0} f_{\tau}(x) \left(\ell_{\tau}(F_{\tau}(\Phi_s(x))) - \ell_{\tau}(F_{\tau}(\Phi_{-s}(x))) \right).$$

We conclude Chapter 5 in Section 5.7 by adapting the conservation law (1.2) to the regularity assumptions in Chapter 5 (Proposition 5.7.1). It turns out that the conservation law is broken by the appearance of a term with varying sign on the right hand side of (1.2). However, if symmetric derivatives vanish macroscopically, the additional term vanishes (Proposition 5.7.2), showing that under this assumption, (1.2) (and hence also current conservation and conservation of energy-momentum) hold macroscopically but not microscopically. This result can be interpreted as giving further support to the assumption of symmetric derivatives to vanish macroscopically.

The results of Chapter 5 can be applied to the setting causal fermion systems as explained in Section 4.1.4.

We expect that the correction terms in the full non-differentiable field equations to second order yield modifications of the field equations in the continuum limit, thus possibly opening the doors to experimental predictions.

1.4 Connection to Foundations of Quantum Theory

As mentioned above, in the so-called continuum limit [Fin16b, Fin14], the Euler-Lagrange equations of the causal action principle give rise to the fundamental equations of quantum theory, general relativity and quantum field theory. The analysis of the jet-formalism in the continuum limit is still open (cf. Chapter 6). However, we have strong reasons for the following conjecture.

Conjecture. In the continuum limit, the linearized field equations (1.7) yield the fundamental equations of contemporary physics, in particular the Dirac equation.

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In this section we evaluate the consequences of this conjecture with respect to quantum theory. Since the Dirac equation reduces to the Pauli equation or the Schrödinger equation in the non-relativistic limit (cf. [BD64, PS95]), the conjecture thus implies that the full non-differentiable field equations to second order (1.16) give rise to correction terms for the Dirac, Pauli and Schrödinger equation.

It is well-known in the foundations of quantum theory that the linearity of the Schrödinger equation conflicts with the von Neumann collapse postulate if one assumes that measurement apparati are composed of objects (“atoms”) which themselves follow the laws of quantum theory. This problem, referred to as the *measurement problem* (cf. [Per93]), can be remedied in several ways, leading to modifications of the original quantum mechanics as formulated by von Neumann [vN32]. Since the theory of causal fermion systems is a candidate for a unified physical theory, this raises the question of what its implications are on the measurement problem.

Equation (1.16) shows that to second order, the corrections for the field equations from the theory of causal fermion systems consist of a stochastic term and a quadratic term. Thus, based on the above conjecture we expect that if one evaluates (1.16) in the continuum limit, the Dirac equation (and hence also the Pauli and Schrödinger equation) arise equipped with an additional stochastic and an additional quadratic term.

This is similar to modifications of the Schrödinger equation which are referred to as spontaneous localization or dynamical collapse models (see [Pea89, GRW86, BLS⁺13, Tum06] or [JZK⁺03, Chapter 8] for a small sample of the vast literature on this topic). In those models, one adds a specific quadratic and a specific stochastic term to the Schrödinger equation in order to break the linearity in a way which is compatible with the experimental observations to date. Thus, according to the above conjecture, the theory of causal fermion systems seems to be an effective dynamical collapse theory, where the term “effective” points to the fact that the theory of causal fermion systems only appears to be such a model in the continuum limit. The fundamental dynamics is yet different from the modified Schrödinger/Pauli/Dirac equation.

Clearly, it remains open at this point which form the correction terms in (1.16) take if evaluated in the continuum limit, and how exactly they compare with the above-mentioned models. But this analysis is to be carried out in the near future. An answer will insofar be interesting as the theory of causal fermion systems and the jet-formalism are completely covariant, i.e. they do not depend on any choice of coordinates on \mathcal{F} (cf. also Section 2.8.6). Hence, if one avoids to break general covariance when taking the limit, the correction terms for the Dirac equation necessarily are covariant as well.

Concerning the conservation laws of Chapter 3 and the generalization to the non-differentiable setting in Section 5.7, we remark the following. Suppose that space-time M is a globally hyperbolic manifold so that we can sensibly talk about “times”. Assume that the wave function undergoes a collapse at some time t_c . It is a reasonable assumption that the continuum limit should still be a good description at some earlier time $t_0 < t_c$ and some later time $t_1 > t_c$. In this situation the conservation law of Theorem 5.7.1 states that the current integrals at times t_0 and t_1 do not coincide if t_0 and t_1 are close to each other. The sign-varying contributions from the symmetric directional derivatives

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$\frac{1}{2}(D_w^+ - D_-^+)\ell(x)$, integrated over the time-strip from t_0 to t_1 destroy current conservation as well as conservation of energy-momentum.

But, if Definition 5.3.2 (‘symmetric derivatives vanish macroscopically’) holds, Proposition 5.7.2 implies that if the time-strip from t_0 to t_1 is a macroscopic region, the collapse mechanism necessarily preserves the normalization of the wave function. Thus, in contrast to some continuous dynamical localization models, given Assumption 5.3.2, in our approach it does not seem to be necessary to rescale the wave function so as to arrange its proper normalization, and similarly for the conservation of energy-momentum in the collapse process.

Clearly, even so our results suggest a particular resolution of the measurement problem based on the theory of causal fermion systems, there remain many open questions about the relation of the theory to foundations of quantum mechanics. (A summary of how entanglement and non-locality arise in this context is given in Section 2.7.) For an overview over the most relevant problems in this context, see the Chapter 6. We plan to carry out further investigations of those and related matters in the near future.

Chapter 2

Causal Fermion Systems as a Candidate for a Unified Physical Theory

This chapter represents an introduction to causal fermion systems which is intended to explain the basic concepts and the general physical picture behind the theory in an easily accessible way. In order to achieve this goal, we avoid technical details if they are not crucial for understanding and explain even simple notions which are otherwise assumed to be known by the reader of this thesis. A thorough introduction to the mathematical setup of causal fermion systems which is relevant for Chapters 3, 4 and 5 can be found in Section 3.3.1 and in [Fin16b, Chapter 1].

This chapter is organized as follows. In Section 2.1 we define the basic objects of the theory. In Section 2.2 we proceed by explaining how those objects appear naturally in the familiar physical situation of Dirac particles in Minkowski space. In Section 2.3 it is shown how the objects of quantum mechanics are encoded in a causal fermion system. Section 2.4 explains for the example of the Minkowski vacuum how a causal fermion system encodes causal structure. In Section 2.5 we exemplify how to describe other physical situations or more general space-times. In Section 2.6 we outline a limiting case in which the causal fermion system can be described by a second-quantized Dirac field coupled to classical gauge fields and gravity. Section 2.7 adds some remarks on the resulting perspective on foundations of quantum mechanics concerning non-locality and entanglement (for remarks about the measurement problem, see Section 1.4). In Section 2.8 we conclude with a few clarifying remarks. This chapter has been published in a slightly modified form as [FK15].

2.1 The Theory

The general structure of the theory of causal fermion systems can be understood in analogy to general relativity. In general relativity, our universe is described by a four-dimensional space-time (Lorentzian manifold) together with particles and fields. However, not every configuration of Lorentzian metric, particles and fields is considered to be “physical” in the sense that it could be realized in nature. Namely, for the configuration to be physically realizable, the Einstein equations must hold. Moreover, the particles must

satisfy the equations of motion, and the additional fields must obey the field equations (like Maxwell's equations). This means that in general relativity, there are two conceptual parts: on the one hand one has mathematical objects describing possible configurations, and on the other hand there is a principle which singles out the physical configurations.

The theory of causal fermion systems has the same conceptual structure consisting of mathematical objects and a principle which singles out the physical configurations. We first introduce the mathematical objects:

Definition 2.1.1. (Causal fermion system)

- ▶ Let $(\mathcal{H}, \langle \cdot | \cdot \rangle_{\mathcal{H}})$ be a separable complex Hilbert space.
- ▶ Given a parameter $n \in \mathbb{N}$ (the *spin dimension*), let $\mathcal{F} \subset L(\mathcal{H})$ be the set of all self-adjoint operators on \mathcal{H} of finite rank, which (counting multiplicities) have at most n positive and at most n negative eigenvalues.
- ▶ Let ρ be a positive measure on \mathcal{F} (the *universal measure*).

Then $(\mathcal{H}, \mathcal{F}, \rho)$ is a *causal fermion system*.

Here separable means that the Hilbert space has an at most countable orthonormal basis. Mapping the basis vectors to each other, one sees that any two Hilbert spaces are isomorphic, provided that their dimensions coincide. Therefore, the structure $(\mathcal{H}, \mathcal{F})$ is completely determined by the parameters $n \in \mathbb{N}$ and $f := \dim \mathcal{H} \in \mathbb{N} \cup \{\infty\}$. Apart from these parameters, the only object specifying a causal fermion system is the universal measure ρ .

It will be outlined below that this definition indeed generalizes mathematical structures used in contemporary physics. The picture is that one causal fermion system describes a space-time together with all structures and objects therein (including the metric, particles and fields).

Next, we state the principle which singles out the physical configurations. Similar to the Lagrangian formulation of contemporary physics, we work with a variational principle, referred to as the *causal action principle*. It states that a causal fermion system which can be realized in nature should be a minimizer of the so-called causal action. In order to formulate the causal action principle, we assume that the Hilbert space $(\mathcal{H}, \langle \cdot | \cdot \rangle_{\mathcal{H}})$ and the spin dimension n have been chosen. Let \mathcal{F} be as in Definition 2.1.1 above. Then for any $x, y \in \mathcal{F}$, the product xy is an operator of rank at most $2n$. We denote its non-trivial eigenvalues (counting algebraic multiplicities) by $\lambda_1^{xy}, \dots, \lambda_{2n}^{xy} \in \mathbb{C}$. We introduce the *spectral weight* $|\cdot|$ of an operator as the sum of the absolute values of its eigenvalues. In particular, the spectral weight of the operator products xy and $(xy)^2$ is defined by

$$|xy| = \sum_{i=1}^{2n} |\lambda_i^{xy}| \quad \text{and} \quad |(xy)^2| = \sum_{i=1}^{2n} |\lambda_i^{xy}|^2.$$

2 Causal Fermion Systems as a Candidate for a Unified Physical Theory

Next, the *Lagrangian* $\mathcal{L} : \mathcal{F} \times \mathcal{F} \rightarrow \mathbb{R}_0^+$ is defined by

$$\mathcal{L}(x, y) := |(xy)^2| - \frac{1}{2n}|xy|^2 = \frac{1}{4n} \sum_{i,j=1}^{2n} \left(|\lambda_i^{xy}| - |\lambda_j^{xy}| \right)^2. \quad (2.1)$$

The particular form of this Lagrangian is the result of research carried out over several years (see Section 2.8.2).

Definition 2.1.2. (Causal action principle) The *causal action* \mathcal{S} is obtained by integrating the Lagrangian with respect to the universal measure,

$$\mathcal{S}(\rho) = \iint_{\mathcal{F} \times \mathcal{F}} \mathcal{L}(x, y) d\rho(x) d\rho(y).$$

The *causal action principle* is to minimize \mathcal{S} under variations of the universal measure, taking into account the following constraints:

$$\text{volume constraint:} \quad \rho(\mathcal{F}) = \text{const} \quad (2.2)$$

$$\text{trace constraint:} \quad \int_{\mathcal{F}} \text{tr}(x) d\rho(x) = \text{const} \quad (2.3)$$

$$\text{boundedness constraint:} \quad \mathcal{T} := \iint_{\mathcal{F} \times \mathcal{F}} |xy|^2 d\rho(x) d\rho(y) \leq C, \quad (2.4)$$

where C is a given constant (and tr denotes the trace of a linear operator on \mathcal{H}).

In mathematical terms, the measure ρ is varied within the class of positive regular Borel measures on \mathcal{F} , where on \mathcal{F} one takes the topology induced by the sup-norm on $L(\mathcal{H})$ (for basic definitions see for example [Rud87, Chapters 2 and 5] or [Hal74, Chapter X]). The volume and trace constraints are needed in order to avoid trivial minimizers and are important for the analysis of the corresponding Euler-Lagrange equations because they give rise to Lagrange multiplier terms. The boundedness constraint is needed in order to ensure the existence of minimizers. In most applications, it does not give rise to a Lagrange multiplier term. Therefore, it does not seem to have any physical consequences.

This concludes the outline of the mathematical definition of the theory. In order to obtain a physical theory, we need to give the mathematical objects a physical interpretation. It is one of the main objectives of the next sections to do so by explaining how the above mathematical objects relate to the common notions in physics. The conclusion will be that causal fermion systems are indeed a candidate for a fundamental physical theory.

2.2 Example: Dirac Wave Functions in Minkowski Space

As a first step towards explaining how causal fermion systems relate to contemporary physics, we now explain how the familiar physical situation of Dirac particles in Minkowski space can be described by a causal fermion system.

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Let \mathcal{M} be Minkowski space and μ the natural volume measure thereon, i.e. $d\mu = d^4x$ if $x = (x^0, x^1, x^2, x^3)$ is an inertial frame (we use the signature convention $(+, -, -, -)$). We consider a finite number of f Dirac particles described by one-particle wave functions ψ_1, \dots, ψ_f which are solutions of the Dirac equation,

$$(i\gamma^j \partial_j - m) \psi_k = 0, \quad k = 1, \dots, f, \quad (2.5)$$

where m is the rest mass, and γ^j are Dirac matrices in the Dirac representation. For simplicity, we assume that the wave functions ψ_1, \dots, ψ_f are continuous.

Before going on, we remark that this description of the f -particle system by f one-particle wave functions departs from the usual Fock space description. The connection to Fock spaces will be explained later in this chapter (see Section 2.7). For the moment, it is preferable to work with the one-particle wave functions. We also remark that the assumption of considering a finite number of continuous wave functions merely is a technical simplification for our presentation. All constructions can be extended to an infinite number of possibly discontinuous wave functions (for details see [FR15, Section 4] or [Fin16b, Chapter 1]).

The wave functions ψ_k span a vector space which we denote by \mathcal{H} ,

$$\mathcal{H} := \text{span}(\psi_1, \dots, \psi_f). \quad (2.6)$$

On \mathcal{H} we consider the usual scalar product on solutions of the Dirac equation

$$\langle \psi | \phi \rangle_{\mathcal{H}} := 2\pi \int_{t=\text{const}} (\bar{\psi} \gamma^0 \phi)(t, \vec{x}) d^3x \quad (2.7)$$

(here $\bar{\psi} = \psi^\dagger \gamma^0$ is the adjoint spinor, where the dagger denotes complex conjugation and transposition). If one evaluates (2.7) for $\phi = \psi$, the integrand can be written as $(\bar{\psi} \gamma^0 \psi)(t, \vec{x}) = (\psi^\dagger \psi)(t, \vec{x})$, having the interpretation as the probability density of the Dirac particle corresponding to ψ to be at the position \vec{x} . In view of the conservation of probability (being a consequence of current conservation), the integral in (2.7) is time independent. Since the probability density is positive, the inner product (2.7) is indeed positive definite. We thus obtain an f -dimensional Hilbert space $(\mathcal{H}, \langle \cdot | \cdot \rangle_{\mathcal{H}})$.

For any $x \in \mathcal{M}$, we now introduce the sesquilinear form

$$b_x : \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{C}, \quad b_x(\psi, \phi) = -(\bar{\psi} \phi)(x),$$

which maps two solutions of the Dirac equation to their inner product at x . The sesquilinear form b_x can be represented by a self-adjoint operator $F(x)$ on \mathcal{H} , which is uniquely defined by the relations

$$\langle \psi | F(x) \phi \rangle_{\mathcal{H}} = b_x(\psi, \phi) \quad \text{for all } \psi, \phi \in \mathcal{H}.$$

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More concretely, in the basis $(\psi_k)_{k=1,\dots,f}$ of \mathcal{H} , the last relation can be written as

$$\langle \psi_i | F(x) \psi_j \rangle_{\mathcal{H}} = -(\overline{\psi_i} \psi_j)(x). \quad (2.8)$$

If the basis is orthonormal, the calculation

$$F(x) \psi_j = \sum_{i=1}^f \langle \psi_i | F(x) \psi_j \rangle_{\mathcal{H}} \psi_i = - \sum_{i=1}^f (\overline{\psi_i} \psi_j)(x) \psi_i$$

(where we used the completeness relation $\phi = \sum_i \langle \psi_i | \phi \rangle \psi_i$), shows that the operator $F(x)$ has the matrix representation

$$(F(x))_j^i = -(\overline{\psi_i} \psi_j)(x).$$

In physical terms, the matrix element $-(\overline{\psi_i} \psi_j)(x)$ gives information on the correlation of the wave functions ψ_i and ψ_j at the space-time point x . Therefore, we refer to $F(x)$ as the *local correlation operator* at x .

Let us analyze the properties of $F(x)$. First of all, the calculation

$$\langle F(x) \psi | \phi \rangle_{\mathcal{H}} = \overline{\langle \phi | F(x) \psi \rangle_{\mathcal{H}}} = -\overline{(\overline{\phi} \psi)(x)} = -(\overline{\psi} \phi)(x) = \langle \psi | F(x) \phi \rangle_{\mathcal{H}}$$

shows that the operator $F(x)$ is self-adjoint (where we denoted complex conjugation by a bar). Furthermore, since the pointwise inner product $(\overline{\psi} \phi)(x)$ has signature $(2, 2)$, we know that b_x has signature (p, q) with $p, q \leq 2$. As a consequence, the operator $F(x)$ has at most two positive and at most two negative eigenvalues (counting multiplicities). It follows immediately, that $F(x) \in \mathcal{F}$ if the spin dimension in Definition 2.1.2 is chosen as $n = 2$.

Constructing the operator $F(x) \in \mathcal{F}$ for every space-time point $x \in M$, we obtain the mapping

$$F : \mathcal{M} \rightarrow \mathcal{F}, \quad x \mapsto F(x).$$

This allows us to introduce a measure ρ on \mathcal{F} as follows. For any $\Omega \subset \mathcal{F}$, one takes the pre-image $F^{-1}(\Omega) \subset \mathcal{M}$ and computes its space-time volume,

$$\rho(\Omega) := \mu(F^{-1}(\Omega)).$$

This gives rise to the so-called *push-forward measure* which in mathematics is denoted by $\rho = F_* \mu$ (see for example [Bog07, Section 3.6]; we remark for the mathematically oriented reader that the σ -algebra of ρ -measurable sets is defined as all sets $\Omega \subset \mathcal{F}$ whose pre-image $F^{-1}(\Omega)$ is μ -measurable).

Putting the above structures together, we obtain a causal fermion system $(\mathcal{H}, \mathcal{F}, \rho)$ of spin dimension two. Thus we have succeeded in constructing a causal fermion system starting from a system of Dirac wave functions in Minkowski space. But it is not obvious how much of the information on the physical system is encoded in the causal fermion

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system. In other words, taking the causal fermion system $(\mathcal{H}, \mathcal{F}, \rho)$ as the starting point, the question is which structures of the original system can be recovered. For example, is the Minkowski metric still determined? Is it possible to reconstruct the Dirac wave functions? Precise answers to these questions will be given in Section 2.3 below. In preparation, we now give a few hints.

We first explain what the points of Minkowski space correspond to in our causal fermion system. Recall that to every space-time point $x \in \mathcal{M}$ we associated a linear operator $F(x) \in \mathcal{F}$. Hence the space-time points correspond to the subset $F(\mathcal{M}) \subset \mathcal{F}$. This subset can also be characterized as the set where the measure ρ is non-zero. In mathematical terms, this is captured in the notion of the *support* of the universal measure, defined as the set of all the points of \mathcal{F} such that every open neighborhood of this point has a non-zero measure. Then (for details see [Fin16b, Chapter 1])

$$\text{supp } \rho = \overline{F(\mathcal{M})}, \quad (2.9)$$

where the bar denotes the closure. In all situations of physical interest, the mapping F will be injective and its image closed (see again [Fin16b, Chapter 1]). Provided that this is the case, identifying $x \in \mathcal{M}$ with the corresponding operator $F(x) \in \mathcal{F}$ makes it possible to identify Minkowski space with the support of ρ as a topological space. Under suitable smoothness and non-degeneracy assumptions, one can identify \mathcal{M} with $\text{supp } \rho$ even as a differentiable manifold. We make this identification manifest by using the letter x for the operator $F(x)$. In order to avoid confusion, we use two different fonts in this chapter, making it possible for the reader to distinguish a point $x \in \mathcal{M}$ of Minkowski space from the corresponding point $x \in M := \text{supp } \rho$. Once the reader has become familiar with our concepts, the different fonts will be unnecessary.

This consideration shows that the topological and differentiable structures of our space-time are encoded in the causal fermion system. Clearly, Minkowski space also has metric and causal structures, which we have not yet addressed. The general idea for recovering these structures is to take operators $x, y \in \text{supp } \rho$ and to analyze the eigenvalues of the operator product xy . The eigenvalues of such operator products contain plenty of information, inducing relations and structures between the space-time points. This will be explained more concretely in the next section.

2.3 Inherent Structures

Let $(\mathcal{H}, \mathcal{F}, \rho)$ be a causal fermion system of spin dimension n (see Definition 2.1.1). We now introduce additional objects which will turn out to generalize familiar notions in physics. All of these structures are inherent in the sense that we only use information already encoded in the causal fermion system.

Motivated by the consideration above (see the paragraph before (2.9)), *space-time* M is defined as the support of the universal measure,

$$M := \text{supp } \rho \subset \mathcal{F}.$$

On M we introduce the following notion of causality. Recall that for $x, y \in M$, the product xy is an operator of rank at most $2n$. We again denote its non-trivial eigenvalues (counting algebraic multiplicities) by $\lambda_1^{xy}, \dots, \lambda_{2n}^{xy} \in \mathbb{C}$.

Definition 2.3.1. (Causality) The space-time points x and y are defined to be

- *spacelike* separated if all the λ_j^{xy} have the same absolute value.
- *timelike* separated if the λ_i^{xy} do not all have the same absolute value and are all real.
- *lightlike* separated if the λ_i^{xy} do not all have the same absolute value and are not all real.

This definition is compatible with the causal action in the following sense. If the points x and y are spacelike separated, then all the λ_j^{xy} have the same absolute value, so that the Lagrangian $\mathcal{L}(x, y)$ vanishes according to (2.1). In a more physical language, this means that no interaction takes place between regions with spacelike separation (this does not exclude nonlocal correlations and entanglement, as will be discussed in Section 2.7). In this way, our setting incorporates a general version of the principle of causality.

The next step is to introduce wave functions. The construction is guided by the usual structure of a Dirac wave function ψ , which to every space-time point x associates a spinor $\psi(x)$. The latter is a vector in the corresponding spinor space $S_x \mathcal{M} \simeq \mathbb{C}^4$, which is endowed with the inner product $\bar{\psi}\phi$ of signature $(2, 2)$. In the setting of causal fermion systems, for a space-time point $x \in M$ we define the *spin space* $S_x \subset \mathcal{H}$ as the image of the operator x ,

$$S_x := x(\mathcal{H}) .$$

It is a subspace of \mathcal{H} of dimension at most $2n$. On S_x we introduce the inner product

$$\prec \cdot | \cdot \succ_x : S_x \times S_x \rightarrow \mathbb{C} , \quad \prec u | v \succ_x := -\langle u | x v \rangle_{\mathcal{H}} , \quad (2.10)$$

referred to as the *spin scalar product*. Since x has at most n positive and at most n negative eigenvalues, the spin scalar product is an indefinite inner product of signature (p, q) with $p, q \leq n$. A *wave function* ψ is defined as a function which to every $x \in M$ associates a vector of the corresponding spin space,

$$\psi : M \rightarrow \mathcal{H} \quad \text{with} \quad \psi(x) \in S_x \quad \text{for all } x \in M .$$

Clearly, it is not sufficient to define wave functions abstractly, but we need to specify those wave functions which are realized in the physical system. Using a familiar physical language, we need to declare which one-particle states are occupied (for the connection to multi-particle Fock states see Section 2.7). To this end, to every vector $u \in \mathcal{H}$ of the Hilbert space we associate a wave function ψ^u by projecting the vector u to the spin spaces, i.e.

$$\psi^u : M \rightarrow \mathcal{H} , \quad \psi^u(x) := \pi_x u \in S_x , \quad (2.11)$$

where π_x is the orthogonal projection in \mathcal{H} on the subspace $x(\mathcal{H}) \subset \mathcal{H}$. We refer to ψ^u as the *physical wave function* corresponding to the vector $u \in \mathcal{H}$.

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Finally, we define the *kernel of the fermionic projector* $P(x, y)$ for any $x, y \in M$ by

$$P(x, y) = \pi_x y|_{S_y} : S_y \rightarrow S_x \quad (2.12)$$

(where $|_{S_y}$ denotes the restriction to the subspace $S_y \subset \mathcal{H}$). This object is useful for analyzing the relations and structures between space-time points. In particular, the kernel of the fermionic projector encodes the causal structure and makes it possible to compute the eigenvalues $\lambda_1^{xy}, \dots, \lambda_{2n}^{xy}$ which appear in the Lagrangian (2.1). In order to see how this comes about, we first define the *closed chain* as the product

$$A_{xy} = P(x, y) P(y, x) : S_x \rightarrow S_x. \quad (2.13)$$

Computing powers of the closed chain and using that $y\pi_y = y$ (because the image and kernel of self-adjoint operators are orthogonal), we obtain

$$A_{xy} = (\pi_x y)(\pi_y x)|_{S_x} = \pi_x yx|_{S_x} \quad \text{and thus} \quad (A_{xy})^p = \pi_x (yx)^p|_{S_x}.$$

Taking the trace, we obtain for all $p \in \mathbb{N}$,

$$\begin{aligned} \text{Tr}_{S_x} ((A_{xy})^p) &= \text{Tr}_{S_x} (\pi_x (yx)^p|_{S_x}) = \text{tr} (\pi_x (yx)^p|_{S_x}) \\ &= \text{tr} ((yx)^p \pi_x) = \text{tr} ((yx)^p) = \text{tr} ((xy)^p) \end{aligned}$$

(where tr again denotes the trace of a linear operator on \mathcal{H}). Since the coefficients of the characteristic polynomial of an operator can be expressed in terms of traces of powers of the corresponding matrix, we conclude that the eigenvalues of the closed chain coincide with the non-trivial eigenvalues $\lambda_1^{xy}, \dots, \lambda_{2n}^{xy}$ of the operator xy in Definition 2.1.2. In this way, one can recover the $\lambda_1^{xy}, \dots, \lambda_{2n}^{xy}$ as the eigenvalues of a $(2n \times 2n)$ -matrix. In particular, the kernel of the fermionic operator encodes the causal structure of M .

The kernel of the fermionic projector is the starting point for constructions which unveil the geometric structures of a causal fermion system. More specifically, this kernel gives rise to a spin connection and corresponding curvature. Moreover, one can introduce tangent spaces endowed with a Lorentzian metric together with a corresponding metric connection and curvature. For brevity, we cannot enter these topics here. Instead we refer the interested reader to [FG12, FK14], where also questions concerning the topology of causal fermion systems are treated. The important point to keep in mind is that all these constructions are tailored in order to understand the meaning of information contained in the causal fermion system. No additional input is required. The system is completely determined by the causal fermion system $(\mathcal{H}, \mathcal{F}, \rho)$. In particular, when varying the universal measure in the causal action principle, one also varies all the derived structures mentioned above.

2.4 The Minkowski Vacuum

In order to illustrate the above inherent structures, we now return to the example of Dirac particles in Minkowski space introduced in Section 2.2. In this example, the Hilbert space \mathcal{H} is spanned by solutions of the Dirac equation. Thus a vector $u \in \mathcal{H}$ is a Dirac wave function, which at a point $x \in \mathcal{M}$ of Minkowski space takes values in the corresponding spinor space, $u(x) \in S_x \mathcal{M}$. On the other hand, in the previous section we introduced the corresponding physical wave function ψ^u , which at a point $x = F(x) \in M \subset \mathcal{F}$ takes values in the corresponding spin space, $\psi^u(x) \in S_x$. We now show that these objects can be identified. Indeed, for any $u, v \in S_x \subset \mathcal{H}$,

$$\prec \psi^u(x) | \psi^v(x) \succ_x \stackrel{(2.10)}{=} -\langle \pi_x u | x \pi_x v \rangle_{\mathcal{H}} = -\langle u | x v \rangle_{\mathcal{H}} = -\langle u | F(x) v \rangle_{\mathcal{H}} \stackrel{(2.8)}{=} \overline{u(x)} v(x) .$$

This shows that the inner products on $S_x \mathcal{M}$ and S_x are compatible. It implies that, after choosing suitable bases, one can indeed identify $S_x \mathcal{M}$ with S_x (for details see [Fin16b, Section 1.2] or [FG12, Section 4]). This identification implies that $\psi^u(x) = u(x)$ for all $u \in \mathcal{H}$ and $x \in M$ respectively $x \in \mathcal{M}$.

Next, it is instructive to bring the kernel of the fermionic projector (2.12) into a more tractable form. To this end, we choose an orthonormal basis u_1, \dots, u_f of \mathcal{H} . Then for any $\phi \in S_y$,

$$\begin{aligned} P(x, y) \phi &= \pi_x y \phi \stackrel{(\star)}{=} \sum_{\ell=1}^f (\pi_x u_\ell) \langle u_\ell | y \phi \rangle_{\mathcal{H}} \\ &\stackrel{(2.10)}{=} - \sum_{\ell=1}^f (\pi_x u_\ell) \prec \pi_y u_\ell | \phi \succ_y \stackrel{(2.11)}{=} - \sum_{\ell=1}^f \psi^{u_\ell}(x) \prec \psi^{u_\ell}(y) | \phi \succ_y , \end{aligned}$$

where in (\star) we used the completeness relation. Using the above identifications of spinors and their inner products, we can write this formula in the shorter form

$$P(x, y) = - \sum_{\ell} u_\ell(x) \overline{u_\ell(y)} . \quad (2.14)$$

This shows that the kernel of the fermionic projector is composed of all the physical wave functions of the system.

In order to work in a more concrete example, we next consider the *Minkowski vacuum*. To this end, we want to implement the concept of the Dirac sea which in non-technical terms states that in the vacuum all the negative-energy states of the Dirac equation should be occupied (see Section 2.8.4 for further explanations of this point). In order to implement this concept, one needs to consider an infinite number of physical wave functions. This can be achieved simply by letting \mathcal{H} in Definition 2.1.1 be an infinite-dimensional Hilbert space. However, a difficulty arises in the construction of the local correlation operators, because the Dirac wave functions (being square-integrable functions) are in general not defined pointwise, so that the right side of (2.8) is ill-defined. In order to resolve this

problem, one needs to introduce an ultraviolet regularization. For conceptual clarity, we postpone the explanation of the ultraviolet regularization to Section 2.6 and now merely mention that an ultraviolet regularization amounts to modifying the Dirac wave functions on a microscopic scale ε , which can be thought of as the Planck scale. In order to avoid the technical issues involved in the regularization, we here simply use the formula (2.14), but now sum over all negative-energy solutions of the Dirac equation. This sum can be rewritten as an integral over the lower mass shell (see again [Fin16b, Section 1.2]),

$$P(x, y) = \int \frac{d^4 k}{(2\pi)^4} (k_j \gamma^j + m) \delta(k^2 - m^2) \Theta(-k_0) e^{-ik(x-y)}. \quad (2.15)$$

In this formula, the necessity for an ultraviolet regularization is apparent in the fact that the Fourier integral is not defined pointwise, but only in the distributional sense. More precisely, the distribution $P(x, y)$ is singular if the vector $\xi := y - x$ is lightlike, but it is a smooth function otherwise (as can be verified for example by explicit computation). As a consequence, a typical ultraviolet regularization will affect the behavior of $P(x, y)$ only in a small neighborhood of the light cone of the form $||\xi^0| - |\vec{\xi}|| \lesssim \varepsilon$. With this in mind, for the following argument we may disregard the ultraviolet regularization simply by restricting attention to the region outside this neighborhood.

The representation (2.15) allows us to understand the relation between the Definition 2.3.1 and the usual notion of causality in Minkowski space: Since the expression (2.15) is Lorentz invariant and is composed of a vector and a scalar component, the function $P(x, y)$ can be written as

$$P(x, y) = \alpha \xi_j \gamma^j + \beta \mathbb{1}$$

with two complex-valued functions α and β (where again $\xi = y - x$). Taking the conjugate with respect to the spin scalar product, we see that

$$P(y, x) = P(x, y)^* = \bar{\alpha} \xi_j \gamma^j + \bar{\beta} \mathbb{1}.$$

As a consequence,

$$A_{xy} = P(x, y) P(y, x) = a \xi_j \gamma^j + b \mathbb{1}$$

with two real-valued functions a and b given by

$$a = \alpha \bar{\beta} + \beta \bar{\alpha}, \quad b = |\alpha|^2 \xi^2 + |\beta|^2.$$

Applying the formula $(A_{xy} - b\mathbb{1})^2 = a^2 \xi^2 \mathbb{1}$, the roots of the characteristic polynomial of A_{xy} are computed by

$$b \pm \sqrt{a^2 \xi^2}.$$

Thus if the vector ξ is timelike, the term ξ^2 is positive, so that the λ_j are all real. By explicit computation one sees that the coefficients a and b are non-zero (see [Fin16b, Section §1.2.5]), implying that the eigenvalues λ_j do not all have the same absolute value. Conversely, if the vector ξ is spacelike, then the term ξ^2 is negative. Thus the λ_j

form a complex conjugate pair, implying that they all have the same absolute value. We conclude that the notions of spacelike and timelike as defined for causal fermion systems in Definition 2.3.1 indeed agree with the usual notions in Minkowski space. We remark that this simple argument cannot be used for lightlike directions because in this case the distribution $P(x, y)$ is singular, making it necessary to consider an ultraviolet regularization (the reader interested in the technical details is referred to [Fin08]).

To summarize, we have seen that the inherent structures of a causal fermion system give back the usual causal structure if one considers the Dirac sea vacuum in Minkowski space. Indeed, a more detailed analysis reveals that the additional inherent structures mentioned at the end of Section 2.3 also give back the geometric structures of Minkowski space (like the metric and the connection).

2.5 Description of More General Space-Times

The constructions explained above also apply to more general physical situations. First, one can consider systems involving *particles* and *anti-particles* by occupying additional states and removing states from the Dirac sea, respectively. Moreover, our construction also apply in *curved space-time* (see [FR15, FR16]) or in the presence of an *external potential* (see [Fin16b]). In all these situations, the resulting causal fermion systems again encode all the information on the physical system (see [FG12, Fin16b]).

The framework of causal fermion systems also allows to describe generalized space-times, sometimes referred as *quantum space-times*. We now illustrate this concept in the simple example of a *space-time lattice*. Thus we replace Minkowski space by a four-dimensional lattice $\mathcal{M} := (\varepsilon\mathbb{Z})^4$ of lattice spacing ε . Likewise, the volume measure d^4x is replaced by a counting measure μ (thus $\mu(\Omega)$ is equal to the number of lattice points contained in Ω). Restricting the Dirac spinors of Minkowski space to the lattice, one gets a spinor space $S_x\mathcal{M}$ at every point $x \in \mathcal{M}$. Dirac wave functions ψ_1, \dots, ψ_f can again be introduced as mappings which to every $x \in \mathcal{M}$ associate a vector in the corresponding spinor space. These Dirac wave functions can be chosen for example as solutions of a discretized version of the Dirac equation. Again choosing \mathcal{H} as the span of the wave functions (2.6) and choosing a suitable scalar product $\langle \cdot | \cdot \rangle_{\mathcal{H}}$, one defines the local correlation operators again by (2.8). Introducing the universal measure as the push-forward of the counting measure μ , we obtain a causal fermion system $(\mathcal{H}, \mathcal{F}, \rho)$ of spin dimension two. The only difference to the causal fermion system in Minkowski space as constructed in Section 2.2 is that now the universal measure is not a continuous but a discrete measure.

When describing the Dirac sea vacuum on the lattice, the lattice spacing gives rise to a natural ultraviolet regularization on the scale ε . For example, one may consider all plane-wave solutions $\psi(x) \sim e^{ikx}$ of the Dirac equation whose four-momenta lie in the first Brillouin zone, i.e. $-\pi < \varepsilon k_j \leq \pi$ for all $j = 0, \dots, 3$. Then one introduces \mathcal{H} as the Hilbert space generated by all these plane-wave solutions restricted to the lattice.

Other examples of discrete or singular space-times are described in [FK14].

2.6 The Continuum Limit

In the previous sections we saw that a causal fermion system has inherent structures which generalize corresponding notions in quantum theory and relativity. The next task is to analyze the dynamics of these objects as described by the causal action principle. To this end, one considers the Euler-Lagrange (EL) equations corresponding to the causal action. These equations have a mathematical structure which is quite different from conventional physical equations (see [BF14]). Therefore, the main difficulty is to reexpress the EL equations in terms of the inherent structures so as to make them comparable with the equations of contemporary physics. This can indeed be accomplished in the so-called continuum limit. Since the mathematical methods needed for the analysis of the continuum limit go beyond the scope of this introduction (for details see [Fin16b, Fin07a]), here we can only explain the general concept and discuss the obtained results.

We outlined in Sections 2.2 and 2.4 how to describe the Minkowski vacuum by a causal fermion system. Recall that the construction required an *ultraviolet regularization* on a microscopic scale ε . Such a regularization can be performed in many different ways. The simplest method is to smooth out the wave functions on the microscopic scale by convolution with a test function. Another method is to introduce a cutoff in momentum space on the scale ε^{-1} . Alternatively, one can regularize by putting the system on a four-dimensional lattice with lattice spacing ε (for example as explained in Section 2.5 above). It is important to note that each regularization gives rise to a different causal fermion system, describing a physical space-time with a different microstructure. Thus in the context of causal fermion systems, the regularization has a physical significance. The freedom in regularizing reflects our lack of knowledge on the microstructure of physical space-time. When analyzing the EL equations corresponding to the causal action, it is not obvious why the effective macroscopic equations should be independent of the regularization details. Therefore, it is necessary to consider a sufficiently large class of regularizations, and one needs to analyze carefully how the results depend on the regularization. This detailed analysis, referred to as the *method of variable regularization* (for more explanations see [Fin07a, §4.1]), reveals that for a large class of regularizations, the structure of the effective macroscopic equations is indeed independent of the regularization (for details see [Fin16b, Chapters 3-5]).

The *continuum limit* is a method for evaluating the EL equations corresponding to the causal action in the limit $\varepsilon \searrow 0$ when the ultraviolet regularization is removed. The effective equations obtained in this limit can be evaluated conveniently in a formalism in which the unknown microscopic structure of space-time (as described by the regularization) enters only in terms of a finite (typically small) number of so-called *regularization parameters*.

It turns out that the causal fermion system describing the Minkowski vacuum satisfies the EL equations in the continuum limit (for any choice of the regularization parameters). If one considers instead a system involving additional particles and anti-particles, it turns out the EL equations in the continuum limit no longer hold. In order to again satisfy these equations, we need to introduce an interaction. In mathematical terms, this

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means that the universal measure ρ must be modified. Expressed in terms of the inherent structures of a causal fermion system, all the physical wave functions $\psi^{u_k}(x)$ must be changed collectively. The analysis shows that this collective behavior of all physical wave functions (including the states of the Dirac sea) can be described by inserting a potential \mathcal{B} into the Dirac equation (2.5),

$$(i\cancel{\partial} + \mathcal{B} - m) u_k(x) = 0, \quad k = 1, \dots, f \quad (2.16)$$

(where as usual $\cancel{\partial} = \gamma^j \partial_j$). Moreover, the EL equations in the continuum limit are satisfied if and only if the potential \mathcal{B} satisfies field equations. Before specifying these field equations, we point out that in the above procedure, the potential \mathcal{B} merely is a convenient device in order to describe the collective behavior of all physical wave functions. It should not be considered as a fundamental object of the theory. We also note that, in order to describe variations of the physical wave functions, the potential in (2.16) can be chosen arbitrarily. Each choice of \mathcal{B} describes a different variation of the physical wave functions. The EL equations in the continuum limit single out the physically admissible potentials as being those which satisfy the field equations.

In [Fin16b] the continuum limit is worked out in several steps beginning from simple systems and ending with a system realizing the fermion configuration of the standard model. For each of these systems, the continuum limit gives rise to effective equations for second-quantized fermion fields coupled to classical bosonic gauge fields (for the connection to second-quantized bosonic fields see Section 2.8.5 below). To explain the structure of the obtained results, it is preferable to first describe the system modelling the leptons as analyzed in [Fin16b, Chapter 4]. The input to this model is the configuration of the leptons in the standard model without interaction. Thus the fermionic projector of the vacuum is assumed to be composed of three generations of Dirac particles of masses $m_1, m_2, m_3 > 0$ (describing e, μ, τ) as well as three generations of Dirac particles of masses $\tilde{m}_1, \tilde{m}_2, \tilde{m}_3 \geq 0$ (describing the corresponding neutrinos). Furthermore, we assume that the regularization of the neutrinos breaks the chiral symmetry (implying that we only see their left-handed components). We point out that the definition of the model does not involve any assumptions on the interaction.

The detailed analysis in [Fin16b, Chapter 4] reveals that the effective interaction in the continuum limit has the following structure. The fermions satisfy the Dirac equation coupled to a left-handed SU(2)-gauge potential $A_L = (A_L^{ij})_{i,j=1,2}$,

$$\left[i\cancel{\partial} + \begin{pmatrix} A_L^{11} & A_L^{12} U_{\text{MNS}}^* \\ A_L^{21} U_{\text{MNS}} & -A_L^{11} \end{pmatrix} \chi_L - mY \right] \psi = 0,$$

where we used a block matrix notation (in which the matrix entries are 3×3 -matrices). Here mY is a diagonal matrix composed of the fermion masses,

$$mY = \text{diag}(\tilde{m}_1, \tilde{m}_2, \tilde{m}_3, m_1, m_2, m_3), \quad (2.17)$$

and U_{MNS} is a unitary 3×3 -matrix (taking the role of the Maki-Nakagawa-Sakata matrix

in the standard model). The gauge potentials A_L satisfy a classical Yang-Mills-type equation, coupled to the fermions. More precisely, writing the isospin dependence of the gauge potentials according to $A_L = \sum_{\alpha=1}^3 A_L^\alpha \sigma^\alpha$ in terms of Pauli matrices, we obtain the field equations

$$\partial^k \partial_l (A_L^\alpha)^l - \square (A_L^\alpha)^k - M_\alpha^2 (A_L^\alpha)^k = c_\alpha \bar{\psi} (\chi_L \gamma^k \sigma^\alpha) \psi, \quad (2.18)$$

valid for $\alpha = 1, 2, 3$ (for notational simplicity, we wrote the Dirac current for one Dirac particle; for a second-quantized Dirac field, this current is to be replaced by the expectation value of the corresponding fermionic field operators). Here M_α are the bosonic masses and c_α the corresponding coupling constants. The masses and coupling constants of the two off-diagonal components are equal, i.e. $M_1 = M_2$ and $c_1 = c_2$, but they may be different from the mass and coupling constant of the diagonal component $\alpha = 3$. Generally speaking, the mass ratios M_1/m_1 , M_3/m_1 as well as the coupling constants c_1 , c_3 depend on the regularization. For a given regularization, they are computable.

Finally, the model involves a gravitational field described by the Einstein equations

$$R_{jk} - \frac{1}{2} R g_{jk} + \Lambda g_{jk} = \kappa T_{jk}, \quad (2.19)$$

where R_{jk} denotes the Ricci tensor, R is scalar curvature, and T_{jk} is the energy-momentum tensor of the Dirac field. Moreover, κ and Λ denote the gravitational and the cosmological constants, respectively. We find that the gravitational constant scales like $\kappa \sim \delta^2$, where $\delta \geq \varepsilon$ is the length scale on which the chiral symmetry is broken.

In [Fin16b, Chapter 5] a system is analyzed which realizes the configuration of the leptons and quarks in the standard model. The result is that the field equation (2.18) is replaced by field equations for the electroweak and strong interactions after spontaneous symmetry breaking (the dynamics of the corresponding Higgs field has not yet been analyzed). Furthermore, the system again involves gravity (2.19).

A few clarifying remarks are in order. First, the above field equations come with corrections which for brevity we cannot discuss here (see [Fin16b, Sections 3.8, 4.4 and 4.6]). Next, it is worth noting that, although the states of the Dirac sea are explicitly taken into account in our analysis, they do not enter the field equations. More specifically, in a perturbative treatment, the divergences of the Feynman diagram describing the vacuum polarization drop out of the EL equations of the causal action. Similarly, the naive “infinite negative energy density” of the sea drops out of the Einstein equations, making it unnecessary to subtract any counter terms. We finally remark that the only free parameters of the theory are the masses in (2.17) as well as the parameter δ which determines the gravitational constant. The coupling constants, the bosonic masses and the mixing matrices are functions of the regularization parameters which are unknown due to our present lack of knowledge on the microscopic structure of space-time. The regularization parameters cannot be chosen arbitrarily because they must satisfy certain relations. But except for these constraints, the regularization parameters are currently treated as free empirical parameters.

To summarize, the dynamics in the continuum limit is described by Dirac spinors coupled to classical gauge fields and gravity. The effective continuum theory is manifestly covariant under general coordinate transformations. The only limitation of the continuum limit is that the bosonic fields are merely classical. However, as will be briefly mentioned in Section 2.8.5, a detailed analysis which goes beyond the continuum limit gives rise even to second-quantized bosonic fields. Based on these results, the theory of causal fermion systems seems to be a promising candidate for a unified physical theory.

2.7 Entanglement and Nonlocality

For general remarks on the connection of the theory of causal fermion systems to foundations of quantum theory, we refer to Section 1.4. Here, based on the previous explanations, we add some remarks on the connection to *nonlocality* and *entanglement*. Both are experimentally tested features of quantum mechanics which need to be explained by any fundamental theory which gives quantum theory as a limiting case.

To understand the role of nonlocality, one should keep in mind that in a causal fermion system, a fermion is described by a physical wave function $\psi^u(x)$ as defined in (2.11). As in standard quantum mechanics, these wave functions are nonlocal objects spread out in space-time, giving rise to the usual nonlocal correlations for one-particle measurements.

In order to describe entanglement, one needs to work with multi-particle wave functions. The simplest method to obtain the connection to those is to choose an orthonormal basis u_1, \dots, u_f of \mathcal{H} and to form the f -particle Hartree-Fock state

$$\Psi := \psi^{u_1} \wedge \dots \wedge \psi^{u_f} . \quad (2.20)$$

Clearly, the choice of the orthonormal basis is unique only up to the unitary transformations

$$u_i \rightarrow \tilde{u}_i = \sum_{j=1}^f U_{ij} u_j \quad \text{with} \quad U \in \text{U}(f) .$$

Due to the anti-symmetrization, this transformation changes the corresponding Hartree-Fock state only by an irrelevant phase factor,

$$\psi^{\tilde{u}_1} \wedge \dots \wedge \psi^{\tilde{u}_f} = \det U \, \psi^{u_1} \wedge \dots \wedge \psi^{u_f} .$$

Thus the configuration of the physical wave functions can be described by a fermionic multi-particle wave function.

The shortcoming of the above construction is that the Hartree-Fock state (2.20) does not allow for the description of entanglement. But entanglement arises naturally if the effect of *microscopic mixing* is taken into account, as we now briefly outline. Microscopic mixing is based on the observation that the causal action of a Dirac sea configuration is smaller if the physical wave functions have fluctuations on the microscopic scale. To be more precise, one constructs a universal measure ρ which consists of L components, i.e. $\rho = \rho_1 + \dots + \rho_L$. This also gives rise to a decomposition of the corresponding

space-time, i.e. $M = M_1 \cup \dots \cup M_L$ with $M_\ell := \text{supp } \rho_\ell$. Now one considers variations of the measures ρ_ℓ obtained by modifying the phases of the physical wave functions in the sub-space-times M_ℓ . Minimizing the causal action under such variations, one sees that the kernel of the fermionic projector $P(x, y)$ becomes very small if x and y are in different sub-space-times. This effect can be understood similar to a dephasing of the physical wave functions in different sub-space-times.

The resulting space-time M has a structure which cannot be understood classically. One way of visualizing M is that it consists of different global space-times M_ℓ which are interconnected by relations between them. An alternative intuitive picture is to regard M as a single space-time which is “fine-grained” on the microscopic scale by the sub-space-times M_ℓ . For the physical wave functions, the above dephasing effect means that every physical wave function $\psi^u(x)$ has *fluctuations* on the microscopic scale. Moreover, comparing $\psi^u(x)$ and $\psi^u(y)$ for x and y in the same sub-space-time, one finds *nonlocal correlations* on the macroscopic scale. A detailed analysis shows that taking averages over the sub-space-times gives rise to an effective description of the interaction in terms of multi-particle wave functions and Fock spaces (see [Fin14, Sections 5, 6 and 8]). In particular, this gives agreement with the usual description of entanglement.

To summarize, entanglement arises naturally in the framework of causal fermion systems when taking into account the effect of microscopic mixing. The reader who wants to understand the concept of microscopic mixing on a deeper quantitative level is referred to [Fin14]. Further remarks about the connection of microscopic mixing to the conservation laws which are established in Chapter 3 are given in Section 3.7.

2.8 Clarifying Remarks

This section aims to address some of the questions which might have come to the mind of the reader.

2.8.1 Where does the name “causal fermion system” come from?

The term “*causality*” in the name causal fermion system refers to the fact that there are causal relations among the space-time points (see Definition 2.3.1). The causal action is “causal” because it vanishes for space-time points with spacelike separation. In this way, the notion of causality is intimately connected with the framework of causal fermion systems. The term “fermion” refers to the fact that a causal fermion system encodes physical wave functions $\psi^u(x)$ (see (2.11)) which are interpreted as *fermionic* wave functions (like Dirac waves). This interpretation as fermionic wave functions is justified because, rewriting the configuration of the physical wave functions in the Fock space formalism, one obtains a totally anti-symmetric multi-particle state (see (2.20)). *Bosonic* fields appear in the causal fermion systems merely as a device to describe the collective behavior of the fermions (see (2.16)).

2.8.2 Why this form of the causal action principle?

The first attempts to formulate a variational principle in space-time in terms of fermionic wave functions can be found in the unpublished preprint [Fin96]. The variational principle proposed in [Fin07a, Section 3.5] coincides with the causal action principle, except that it is formulated in the setting of discrete space-times and that the constraints (2.2) and (2.3) are missing. The general structure of the Lagrangian (2.1) can be understood from the requirements that it should be non-negative and that it should vanish for spacelike separation. The detailed form of the Lagrangian (2.1) is determined uniquely by demanding that the Dirac sea vacuum should be a stable minimizer of the variational principle (as is made precise by the notion of “state stability”; see [Fin07a, Section 5.6]). The necessity and significance of the constraints (2.2) and (2.3) became clear when analyzing the existence theory [Fin07b, Fin10a] and deriving the EL equations [BF14]. It should also be noted that the so-called identity constraint considered in [Fin10a] has turned out to be a too strong condition which is not compatible with the so-called spatial normalization of the fermionic projector as discussed in [FT14, Section 2.2] and established by Chapter 3 (cf. Remark 3.4.14).

2.8.3 Why the name “continuum limit”?

Causal fermion systems were first analyzed in the more restrictive formulation of discrete space-times (see [Fin07a, Section 3.3]). In this setting, the continuum limit as introduced in [Fin07a, Chapter 4] arises as the limit when the discretization scale ε tends to zero, meaning that the discrete space-time goes over to a space-time continuum. The more general notion of causal fermion systems given here allows for the description of both continuous and discrete space-times. Then the parameter ε should be regarded as a regularization length, but space-time could very well be continuous on this scale. In this more general context, the notion “continuum limit” merely means that we take the limit $\varepsilon \searrow 0$ in which space-time $M := \text{supp } \rho$ goes over to the *usual* space-time continuum \mathcal{M} (i.e. Minkowski space or a Lorentzian manifold).

2.8.4 Connection to the notion of the Dirac sea

The concept of the Dirac sea was introduced by Dirac in order to remedy the problem of the negative-energy solutions of the Dirac equation. Dirac’s original conception was that in vacuum all negative-energy states are occupied. Due to the Pauli exclusion principle, additional particles must occupy states of positive energy. This concept led to the prediction of anti-particles, which are described as “holes” in the sea.

If taken literally, the concept of the Dirac sea leads to problems such as an infinite negative energy density or an infinite charge density. This is the main reason why in modern quantum field theory, the concept of the Dirac sea is no longer apparent. It has been replaced by Wick ordering and the reinterpretation of creation and annihilation operators corresponding to the negative-energy states. Therefore, it is a common view that the Dirac sea is merely a historical relic which is no longer needed.

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In the theory of causal fermion system, Dirac's original concept is revived. Namely, when constructing a causal fermion system starting from a classical space-time the states of the Dirac sea need to be taken into account (cf. (2.14) and (2.15) in the Minkowski vacuum). This can be understood as follows. It is a general concept behind causal fermion systems that all structures in space-time should be encoded in the physical wave functions. This concept only works if there are "sufficiently many" physical wave functions. More specifically, this is the case if the causal fermion system is composed of a regularized Dirac sea configuration, possibly with additional particles and/or anti-particles.

In contrast to the problems in the naive Dirac sea picture, in the description with causal fermion systems the ensemble of the sea states does *not* give rise to an infinite negative energy density or an infinite charge density. Namely, due to the specific form of the causal action principle, the sea states drop out of the Euler-Lagrange equations in the continuum limit.

2.8.5 Connection to quantum field theory

The continuum limit gives an effective description of the interaction on the level of second-quantized fermionic fields coupled to classical bosonic fields. A full quantum field theory, in which also the bosonic fields are quantized, arises if the effect of microscopic mixing is taken into account. We refer the reader to Section 2.7 as well as to [Fin14]. The detailed analysis of the resulting Feynman diagrams, renormalization and a comparison with standard quantum field theory is work in progress.

2.8.6 Which physical principles are incorporated in a causal fermion system?

Causal fermion systems evolved from an attempt to combine several physical principles in a coherent mathematical framework. As a result, these principles appear in the framework in a specific way:

- ▶ The **principle of causality** is built into a causal fermion system in a specific way, as explained in Section 2.8.1 above.
- ▶ The **Pauli exclusion principle** is incorporated in a causal fermion system, as can be seen in various ways. One formulation of the Pauli exclusion principle states that every fermionic one-particle state can be occupied by at most one particle. In this formulation, the Pauli exclusion principle is respected because every wave function can either be represented in the form ψ^u (the state is occupied) with $u \in \mathcal{H}$ or it cannot be represented as a physical wave function (the state is not occupied). But it is impossible to describe higher occupation numbers. When working with multi-particle wave functions, the Pauli exclusion principle becomes apparent in the total anti-symmetrization of the wave function (see (2.20)).
- ▶ A **local gauge principle** becomes apparent once we choose basis representations of the spin spaces and write the wave functions in components. Denoting the signature of (S_x, \prec, \succ_x) by $(p(x), q(x))$, we choose a pseudo-orthonormal basis $(\mathbf{e}_\alpha(x))_{\alpha=1, \dots, p+q}$

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of S_x . Then a wave function ψ can be represented as

$$\psi(x) = \sum_{\alpha=1}^{p+q} \psi^\alpha(x) \mathbf{e}_\alpha(x)$$

with component functions $\psi^1, \dots, \psi^{p+q}$. The freedom in choosing the basis (\mathbf{e}_α) is described by the group $U(p, q)$ of unitary transformations with respect to an inner product of signature (p, q) . This gives rise to the transformations

$$\mathbf{e}_\alpha(x) \rightarrow \sum_{\beta=1}^{p+q} U^{-1}(x)_\alpha^\beta \mathbf{e}_\beta(x) \quad \text{and} \quad \psi^\alpha(x) \rightarrow \sum_{\beta=1}^{p+q} U(x)_\beta^\alpha \psi^\beta(x)$$

with $U \in U(p, q)$. As the basis (\mathbf{e}_α) can be chosen independently at each space-time point, one obtains *local gauge transformations* of the wave functions, where the gauge group is determined to be the isometry group of the spin scalar product. The causal action is *gauge invariant* in the sense that it does not depend on the choice of spinor bases.

- The **equivalence principle** is incorporated in the following general way. Space-time $M := \text{supp } \rho$ together with the universal measure ρ form a topological measure space, being a more general structure than a Lorentzian manifold. Therefore, when describing M by local coordinates, the freedom in choosing such coordinates generalizes the freedom in choosing general reference frames in a space-time manifold. Therefore, the equivalence principle of general relativity is respected. The causal action is *generally covariant* in the sense that it does not depend on the choice of coordinates.

2.8.7 Philosophical remarks

Since causal fermion systems are a candidate for a unified physical theory, one may take a consistent realist point of view and assume that our universe is a causal fermion system. Here by “realist point of view” we mean that one assumes that there is a reality independent of human observation and that one can describe this reality in a mathematical language. “Consistent” means that this point of view does not lead to contradictions or inconsistencies. Finally, by “universe is a causal fermion system” we mean that the fundamental entities of our universe are the causal fermion system $(\mathcal{H}, \mathcal{F}, \rho)$ as well as its inherent structures.

This position could be investigated from a philosophical point of view. We find the following points interesting:

- Space-time is a set of operators. The relations between space-time points are all encoded in properties of products of these operators. No additional structures need to be specified.
- Similar to the picture in dynamical collapse theories, the basic object to describe a fermion is the physical wave function $\psi^u(x)$. The particle character, however, comes

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about merely as a consequence of the dynamics as described by the causal action principle.

- The structures of space-time and matter are described in terms of a single object: the universal measure. In particular, it is no longer possible to separate space-time from the matter content therein. This seems to go a step further than relativity: In relativity, space and time do not exist separately, but are combined to space-time. In the approach of causal fermion systems, space-time does not exist without the matter content (including the Dirac sea). Space-time and the matter content are combined in one object.

A further investigation of these and related points might offer new perspectives on questions in philosophy of physics.

Chapter 3

Noether-Like Theorems

In this chapter, we explore symmetries and the resulting conservation laws in the framework of causal fermion systems. We prove that there are indeed conservation laws, which however have a structure which is quite different from that of the classical Noether theorem. These conservation laws are so general that they apply to “quantum space-times” which cannot be approximated by a Lorentzian manifold. We prove that in the proper limiting case, our conservation laws simplify to charge conservation and the conservation of energy and momentum in Minkowski space.

In order to make this chapter easily accessible and self-contained, we develop our concepts step by step. Section 3.1 provides the necessary background: After a brief review of the classical Noether theorem (Section 3.1.1), we introduce causal variational principles in the compact setting (a mathematical simplification of the setting of causal fermion systems) in Section 3.1.2 and define the concept of surface layer integrals in Section 3.1.3.

After these preparations, in Section 3.2 we prove conservation laws for causal variational principles in the compact setting. We distinguish two different kinds of symmetries: symmetries of the Lagrangian (see Definition 3.2.4 and Theorem 3.2.5) and symmetries of the universal measure (see Definition 3.2.2 and Theorem 3.2.3). These symmetries and the corresponding conservation laws can be combined in so-called *generalized integrated symmetries* (see Definition 3.2.7 and Theorem 3.2.8).

In Section 3.3 we generalize the previous results to the setting of causal fermion systems. After a brief introduction to the mathematical setup (Section 3.3.1), we derive corresponding Noether-like theorems (see Theorem 3.3.7, Corollary 3.3.9 and Corollary 3.3.10 in Section 3.3.2). In the following Sections 3.4 and 3.5, we work out examples which give the correspondence to current conservation (Theorem 3.4.3) and to the conservation of energy-momentum (Corollary 3.5.4) in a limiting case. In Section 3.4.3, the mathematical assumptions and the physical picture is discussed and clarified by a few remarks. In Section 3.6 it is explained why the conservation laws corresponding to symmetries of the universal measure are trivially satisfied in Minkowski space and do not capture any interesting dynamical information. Finally, in Section 3.7, we explain the relation to the mechanism of microscopic mixing of the wave functions (as introduced in [Fin14, Section 3] and briefly explained in Section 2.7). This chapter has been published with minor modifications as [FK16].

3.1 Preliminaries

3.1.1 The Classical Noether Theorem

We now briefly review Noether's theorem [Noe18] in the form most suitable for our purposes (similar formulations are found in [Gol80, Section 13.7] or [Bar80, Chapter III]). For simplicity, we begin in four-dimensional Minkowski space \mathcal{M} . In the Lagrangian formulation of classical field theory, one seeks for critical points of an action of the form

$$\mathcal{S} = \int_{\mathcal{M}} \mathcal{L}(\psi(x), \psi_{,j}(x), x) d^4x$$

(where ψ is for example a scalar, tensor or spinor field, and $\psi_{,j} \equiv \partial_j \psi$ denotes the partial derivative). The critical field configurations satisfy the Euler-Lagrange (EL) equations

$$\frac{\partial \mathcal{L}}{\partial \psi} - \frac{\partial}{\partial x^j} \left(\frac{\partial \mathcal{L}}{\partial \psi_{,j}} \right) = 0. \quad (3.1)$$

Symmetries are formulated in terms of variations of the field and the space-time coordinates. More precisely, for given $\tau_{\max} > 0$ we consider smooth families (ψ_τ) and (x_τ) parametrized by $\tau \in (-\tau_{\max}, \tau_{\max})$ with $\psi_\tau|_{\tau=0} = \psi$ and $x_\tau|_{\tau=0} = x$. We assume that these variations describe a *symmetry of the action*, meaning that for every compact space-time region $\Omega \subset \mathcal{M}$ and every field configuration ψ the equation

$$\int_{\Omega} \mathcal{L}(\psi(x), \psi_{,j}(x), x) d^4x = \int_{\Omega'} \mathcal{L}(\psi_\tau(y), (\psi_\tau)_{,j}(y), y) d^4y \quad (3.2)$$

holds for all $\tau \in (-\tau_{\max}, \tau_{\max})$, where $\Omega' = \{x_\tau \mid x \in \Omega\}$ is the transformed region. The corresponding *Noether current* J is defined by

$$J^k = \frac{\partial \mathcal{L}}{\partial \psi_{,k}} \delta \psi + \mathcal{L} \delta x^k - \frac{\partial \mathcal{L}}{\partial \psi_{,k}} \partial_j \psi \delta x^j,$$

where δx and $\delta \psi$ are the first variations

$$\delta x := \frac{d}{d\tau} x_\tau|_{\tau=0} \quad \text{and} \quad \delta \psi(x) := \frac{d}{d\tau} \psi_\tau(x_\tau)|_{\tau=0}.$$

Noether's theorem states that if ψ satisfies the EL equations, then the Noether current is divergence-free,

$$\partial_k J^k = 0.$$

Using the Gauß divergence theorem, one may integrate this equation to obtain a corresponding conserved quantity. To this end, one chooses a space-time region Ω whose boundary $\partial\Omega$ consists of two space-like hypersurfaces \mathcal{N}_1 and \mathcal{N}_2 . Then

$$\int_{\mathcal{N}_1} J^k \nu_k d\mu_{\mathcal{N}_1} = \int_{\mathcal{N}_2} J^k \nu_k d\mu_{\mathcal{N}_2}, \quad (3.3)$$

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where ν is the future-directed normal, and $d\mu_{\mathcal{N}_{1/2}}$ is the induced volume measure (if Ω is unbounded, one needs to assume suitable decay of J^k at infinity).

We now mention two well-known applications of Noether's theorem which will be most relevant here. The first application is to consider the Lagrangian of a quantum mechanical wave function ψ (like the Schrödinger, Klein-Gordon or Dirac Lagrangian) and to consider global phase transformations of the wave function,

$$\psi_\tau(x) = e^{i\tau} \psi(x), \quad x_\tau = x. \quad (3.4)$$

Then the symmetry condition (3.2) is satisfied because the Lagrangian depends only on the modulus of ψ . The corresponding Noether current is the probability current, giving rise to *current conservation*. We remark that, if the quantum mechanical wave function is coupled to an electromagnetic field, then this current coincides, up to a multiplicative constant, with the electromagnetic current of the particle. Therefore, the conservation law can also be interpreted as the *conservation of electric charge*. The second application is to consider translations in space-time, i.e.

$$\psi_\tau(x) = \psi(x), \quad x_\tau = x + \tau v$$

with a fixed vector $v \in \mathcal{M}$. In this case, the symmetry condition (3.2) is satisfied if we assume that $\mathcal{L} = \mathcal{L}(\phi, \phi_{,j})$ does not depend explicitly on x . After a suitable symmetrization procedure (see [LL62, §32 and §94] or the systematic treatment in [FR04]), the corresponding Noether current can be written as

$$J^k = T^{kj} v_j,$$

where T_{jk} is the energy-momentum tensor. Noether's theorem yields the *conservation of energy and momentum*.

Noether's theorem also applies in curved space-time. In this case, the Lagrangian involves the Lorentzian metric. As a consequence, the symmetry condition (3.2) implies that the metric must be invariant under the variation x_τ . This is made precise by the notion of a *Killing field* K , being a vector field which satisfies the Killing equation

$$\nabla_i K_j = -\nabla_j K_i$$

(see for example [HE73, Section 2.6] or [Str04, Section 1.9]). If space-time admits a Killing field K , the corresponding Noether current is most conveniently constructed as follows. As a consequence of the Einstein equations, the energy-momentum tensor is divergence-free,

$$\nabla_j T^{jk} = 0.$$

This by itself does not give rise to conserved quantities because the Gauß divergence theorem only applies to vector fields, but not to tensor fields. However, a direct computation

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shows that contracting the energy-momentum tensor with the Killing field,

$$J^k := T^{kj} K_j ,$$

gives rise to a divergence-free vector field (see [HE73, Section 3.2] or [Str04, Section 2.4]). Hence integration again gives a conservation law of the form (3.3).

3.1.2 Causal Variational Principles in the Compact Setting

We now introduce the setting of causal variational principles in the compact case, slightly generalizing the presentation in [FS13, Section 1.2]. Let \mathcal{F} be a smooth *compact* manifold and $\mathcal{L} \in C^{0,1}(\mathcal{F} \times \mathcal{F}, \mathbb{R}_0^+)$ a non-negative Lipschitz-continuous function which is symmetric, i.e.

$$\mathcal{L}(x, y) = \mathcal{L}(y, x) \quad \text{for all } x, y \in \mathcal{F} . \quad (3.5)$$

The *causal variational principle* is to minimize the action \mathcal{S} defined by

$$\mathcal{S}(\rho) = \iint_{\mathcal{F} \times \mathcal{F}} \mathcal{L}(x, y) d\rho(x) d\rho(y) \quad (3.6)$$

under variations of ρ in the class of (positive) normalized regular Borel measures. The existence of minimizers follows immediately from abstract compactness arguments (see [Fin10a, Section 1.2]).

In what follows, we let ρ be a given minimizing measure, referred to as the *universal measure*. The resulting EL equations are derived in [FS13, Section 3.1]. For the sake of self-consistency, we now state them and repeat the proof.

Lemma 3.1.1. (Euler-Lagrange equations) *Let ρ be a minimizing measure of the causal variational principle (3.6). Then the function $\ell \in C^{0,1}(\mathcal{F})$ defined by*

$$\ell(x) = \int_{\mathcal{F}} \mathcal{L}(x, y) d\rho(y) \quad (3.7)$$

is minimal on the support of ρ ,

$$\ell|_{\text{supp } \rho} \equiv \inf_{\mathcal{F}} \ell . \quad (3.8)$$

We remark that in Chapters 4 and 5, we add a constant $-\frac{\nu}{2}$ to the right hand side of (3.7) which we choose such that $\inf_{\mathcal{F}} \ell = 0$ (compare e.g. (4.2) and (4.10)). However, in the present context, this additional property of ℓ is not necessary.

Proof. Carrying out one of the integrals, one sees that

$$\mathcal{S}(\rho) = \iint_{\mathcal{F} \times \mathcal{F}} \mathcal{L}(x, y) d\rho(x) d\rho(y) = \int_{\mathcal{F}} \ell d\rho . \quad (3.9)$$

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Since ℓ is continuous and \mathcal{F} is compact, there clearly is $y \in \mathcal{F}$ with

$$\ell(y) = \inf_{\mathcal{F}} \ell .$$

We consider for $\tau \in [0, 1]$ the family of normalized regular Borel measures

$$\tilde{\rho}_\tau = (1 - \tau) \rho + \tau \delta_y ,$$

where δ_y denotes the Dirac measure supported at y . Applying this formula in (3.6) and differentiating, we obtain for the first variation

$$\delta \mathcal{S} := \lim_{\tau \searrow 0} \frac{\mathcal{S}(\tilde{\rho}_\tau) - \mathcal{S}(\tilde{\rho}_0)}{\tau} = -2\mathcal{S}(\rho) + 2\ell(y) .$$

Since ρ is a minimizer, $\delta \mathcal{S}$ is non-negative. Hence

$$\inf_{\mathcal{F}} \ell = \ell(y) \geq \mathcal{S}(\rho) \stackrel{(3.9)}{=} \int_{\mathcal{F}} \ell \, d\rho .$$

It follows that ℓ is constant on the support of ρ , giving the result. \square

As explained in detail in Chapter 2, the physical picture is that the universal measure gives rise to a space-time and also induces all the objects therein. In the compact setting considered here, one only obtains space-time endowed with a causal structure in the following way. Space-time is defined as the support of the universal measure,

$$\text{space-time} \quad M := \text{supp } \rho .$$

For a space-time point $x \in M$, we define the open *light cone* $\mathcal{I}(x)$ and the closed light cone $\mathcal{J}(x)$ by

$$\mathcal{I}(x) = \{y \in M \mid \mathcal{L}(x, y) > 0\} \quad \text{and} \quad \mathcal{J}(x) = \overline{\mathcal{I}(x)} .$$

This makes it possible to define a *causal structure* on space-time by saying that two space-time points $x, y \in M$ are *timelike* separated if $\mathcal{L}(x, y) > 0$ and *spacelike* separated if $\mathcal{L}(x, y) = 0$. We remark that, in the setting of causal fermion systems, these notions indeed agree with the usual notion of causality in Minkowski space or on a globally hyperbolic manifold (cf. Section 2.4 and [Fin16b]).

3.1.3 The Concept of Surface Layer Integrals

It is not at all obvious how the classical Noether theorem should be generalized to causal variational principles. First, the mathematical structure of the EL equations (3.8) is completely different from that of the classical EL equations (3.1). Moreover, for writing down surface integrals as in (3.3) one needs structures like the Lorentzian metric as well as the normal to a hypersurface and the induced volume measure thereon. All

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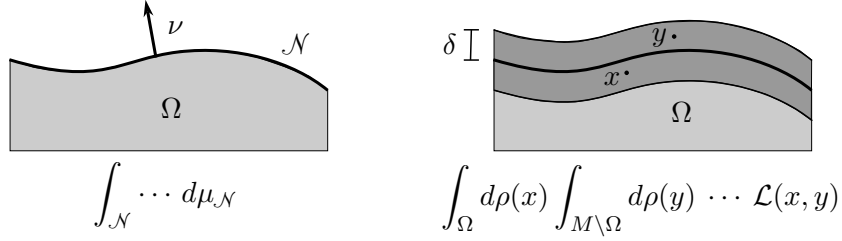


Figure 3.1: A surface integral and a corresponding surface layer integral.

these structures are not directly available in the setting of causal variational principles. Therefore, it is a priori not clear how conservation laws should be stated.

The first task is to introduce an analog of the surface integral in (3.3). The only objects to our disposal are the Lagrangian $\mathcal{L}(x, y)$ and the universal measure ρ . We make the assumption that the Lagrangian is of *short range* in the following sense. We let $d \in C^0(M \times M, \mathbb{R}_0^+)$ be a distance function on M (since M is compact, any two such distance functions are equivalent). The assumption of short range means that \mathcal{L} vanishes on distances larger than δ , i.e.

$$d(x, y) > \delta \implies \mathcal{L}(x, y) = 0 \quad (3.10)$$

Then a double integral of the form

$$\int_\Omega \left(\int_{M \setminus \Omega} \dots \mathcal{L}(x, y) d\rho(y) \right) d\rho(x) \quad (3.11)$$

only involves pairs (x, y) of distance at most δ , where x is in Ω and y is in the complement $M \setminus \Omega$. Thus the integral only involves points in a layer around the boundary of Ω of width δ , i.e.

$$x, y \in B_\delta(\partial\Omega).$$

Therefore, a double integral of the form (3.11) can be regarded as an approximation of a surface integral on the length scale δ , as shown in Figure 3.1. We refer to integrals of the form (3.11) as *surface layer integrals*. In the setting of causal variational principles, they take the role of surface integrals in Lorentzian geometry. Our strategy is to find expressions for the integrand “...” in (3.11) such that the surface layer integral vanishes. Choosing Ω as a space-time region such that $\partial\Omega$ has two connected components \mathcal{N}_1 and \mathcal{N}_2 , one then obtains a conservation law similar to (3.3), with the surface integrals replaced by corresponding surface layer integrals.

We remark for clarity that the correspondence between surface integrals and surface layer integrals could be made mathematically precise by taking the limit $\delta \searrow 0$. However, this would make it necessary to consider a family of Lagrangians \mathcal{L}_δ together with corresponding minimizers ρ_δ . This seems an interesting technical problem for the future. For our purposes, it suffices to identify the surface layer integrals (3.11) as the objects

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which replace the usual surface integrals.

We finally remark that, in the physical setting of causal fermion systems, the condition of short range (3.10) will be replaced by the weaker requirement that the main contribution to the double integral (3.11) comes from pairs of points (x, y) whose distance is at most δ . This will be explained in detail in Section 3.4.2, where we will also identify the length scale δ with the Compton scale (see the paragraph after (3.83)).

3.2 Noether-Like Theorems in the Compact Setting

We now derive Noether-like theorems in the compact setting. We consider two different symmetries: symmetries of the Lagrangian (Theorem 3.2.3) and symmetries of the universal measure (Theorem 3.2.5). In Section 3.2.3, these symmetries will be combined in the notion of generalized integrated symmetries (Theorem 3.2.8).

3.2.1 Symmetries of the Lagrangian

The assumption (3.2) can be understood as a symmetry condition for the Lagrangian. We now want to impose a similar symmetry condition for the Lagrangian $\mathcal{L}(x, y)$ of a causal variational principle. The most obvious method would be to consider a one-parameter group of diffeomorphisms Φ_τ ,

$$\Phi : \mathbb{R} \times \mathcal{F} \rightarrow \mathcal{F} \quad \text{with} \quad \Phi_\tau \Phi_{\tau'} = \Phi_{\tau+\tau'} \quad (3.12)$$

and to impose that \mathcal{L} be invariant under these diffeomorphisms in the sense that

$$\mathcal{L}(x, y) = \mathcal{L}(\Phi_\tau(x), \Phi_\tau(y)) \quad \text{for all } \tau \in \mathbb{R} \text{ and } x, y \in \mathcal{F}. \quad (3.13)$$

However, this condition is unnecessarily strong for two reasons. First, it suffices to consider families which are defined locally for $\tau \in (-\tau_{\max}, \tau_{\max})$. Second, the mapping Φ does not need to be defined on all of \mathcal{F} . Instead, it is more appropriate to impose the symmetry condition only on space-time $M \subset \mathcal{F}$. This leads us to consider instead of (3.12) a mapping

$$\Phi : (-\tau_{\max}, \tau_{\max}) \times M \rightarrow \mathcal{F} \quad \text{with} \quad \Phi(0, \cdot) = \mathbb{1}. \quad (3.14)$$

We also write $\Phi_\tau(x) \equiv \Phi(\tau, x)$ and refer to Φ_τ as a **variation** of M in \mathcal{F} . Next, we need to specify what we mean by “smoothness” of this variation. This is a subtle point because in view of the results in [FS13], the universal measure does not need to be smooth (in the sense that it cannot in general be written as a smooth function times the Lebesgue measure), and therefore the function ℓ will in general only be Lipschitz continuous. Our Noether-like theorems require only that the function ℓ be differentiable in the direction of the variations:

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Definition 3.2.1. A variation Φ_τ of the form (3.14) is **continuously differentiable** if the composition

$$\ell \circ \Phi : (-\tau_{\max}, \tau_{\max}) \times M \rightarrow \mathbb{R}$$

is continuous and if its partial derivative $\partial_\tau(\ell \circ \Phi)$ exists and is continuous.

The next question is how to adapt the symmetry condition (3.13) to the mapping Φ defined only on $(-\tau_{\max}, \tau_{\max}) \times M$. This is not obvious because setting $\tilde{x} = \Phi_\tau(x)$ and using the group property, the condition (3.13) can be written equivalently as

$$\mathcal{L}(\Phi_{-\tau}(\tilde{x}), y) = \mathcal{L}(\tilde{x}, \Phi_\tau(y)) \quad \text{for all } \tau \in \mathbb{R} \text{ and } \tilde{x}, y \in \mathcal{F}. \quad (3.15)$$

But if we restrict attention to pairs $x, y \in M$, the equations in (3.13) and (3.15) are different. It turns out that the correct procedure is to work with the expression in (3.15).

Definition 3.2.2. A variation Φ_τ of the form (3.14) is a **symmetry of the Lagrangian** if

$$\mathcal{L}(x, \Phi_\tau(y)) = \mathcal{L}(\Phi_{-\tau}(x), y) \quad \text{for all } \tau \in (-\tau_{\max}, \tau_{\max}) \text{ and } x, y \in M. \quad (3.16)$$

We now state our first Noether-like theorem.

Theorem 3.2.3. Let ρ be a minimizing measure and Φ_τ a continuously differentiable symmetry of the Lagrangian. Then for any compact subset $\Omega \subset M$, we have

$$\frac{d}{d\tau} \int_{\Omega} d\rho(x) \int_{M \setminus \Omega} d\rho(y) \left(\mathcal{L}(\Phi_\tau(x), y) - \mathcal{L}(\Phi_{-\tau}(x), y) \right) \Big|_{\tau=0} = 0. \quad (3.17)$$

Before coming to the proof, we explain the connection to surface layer integrals. To this end, let us assume that Φ_τ and the Lagrangian are differentiable in the sense that the derivatives

$$\frac{d}{d\tau} \Phi_\tau(x) \Big|_{\tau=0} =: u(x) \quad \text{and} \quad \frac{d}{d\tau} \mathcal{L}(\Phi_\tau(x), y) \Big|_{\tau=0} \quad (3.18)$$

exist for all $x, y \in M$ and are continuous on M respectively $M \times M$. Then one may exchange differentiation and integration in (3.17) and apply the chain rule to obtain

$$\int_{\Omega} d\rho(x) \int_{M \setminus \Omega} d\rho(y) D_{u(x)} \mathcal{L}(x, y) = 0,$$

where $D_{u(x)}$ is the derivative in the direction of the vector field $u(x)$. This expression is a surface layer integral as in (3.11). In general, the derivatives in (3.18) need *not* exist, because we merely imposed the weaker differentiability assumption of Definition 3.2.1. In this case, the statement of the theorem implies that the derivative of the integral in (3.17) exists and vanishes.

Proof of Theorem 3.2.3. We multiply (3.16) by a bounded measurable function f on M

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and integrate. This gives

$$\begin{aligned} 0 &= \iint_{M \times M} f(x) f(y) \left(\mathcal{L}(x, \Phi_\tau(y)) - \mathcal{L}(\Phi_{-\tau}(x), y) \right) d\rho(x) d\rho(y) \\ &= \iint_{M \times M} f(x) f(y) \left(\mathcal{L}(\Phi_\tau(x), y) - \mathcal{L}(\Phi_{-\tau}(x), y) \right) d\rho(x) d\rho(y) , \end{aligned}$$

where in the last step we used the symmetry of the Lagrangian (3.5) and the symmetry of the integrand in x and y . We replace $f(y)$ by $1 - (1 - f(y))$, multiply out and use the definition of ℓ , (3.7). We thus obtain

$$\begin{aligned} 0 &= \int_M f(x) \left(\ell(\Phi_\tau(x)) - \ell(\Phi_{-\tau}(x)) \right) d\rho(x) \\ &\quad - \iint_{M \times M} f(x) (1 - f(y)) \left(\mathcal{L}(\Phi_\tau(x), y) - \mathcal{L}(\Phi_{-\tau}(x), y) \right) d\rho(x) d\rho(y) . \end{aligned}$$

Choosing f as the characteristic function of Ω , we obtain the identity

$$\begin{aligned} &\int_\Omega d\rho(x) \int_{M \setminus \Omega} d\rho(y) \left(\mathcal{L}(\Phi_\tau(x), y) - \mathcal{L}(\Phi_{-\tau}(x), y) \right) \\ &= \int_\Omega \left(\ell(\Phi_\tau(x)) - \ell(\Phi_{-\tau}(x)) \right) d\rho(x) . \end{aligned} \tag{3.19}$$

Using that $\ell(\Phi_\tau(x))$ is continuously differentiable (see Definition 3.2.1) and that Ω is compact, we conclude that the right side of this equation is differentiable at $\tau = 0$. Moreover, we are allowed to exchange the τ -differentiation with integration. The EL equations (3.8) imply that

$$\frac{d}{d\tau} \ell(\Phi_\tau(x)) \Big|_{\tau=0} = 0 = \frac{d}{d\tau} \ell(\Phi_{-\tau}(x)) \Big|_{\tau=0} . \tag{3.20}$$

Hence the right side of (3.19) is differentiable at $\tau = 0$, and the derivative vanishes. This gives the result. \square

3.2.2 Symmetries of the Universal Measure

We now prove a conservation law for a different type of symmetry.

Definition 3.2.4. A variation Φ_τ of the form (3.14) is a **symmetry of the universal measure** if

$$(\Phi_\tau)_* \rho = \rho \quad \text{for all } \tau \in (-\tau_{\max}, \tau_{\max}) . \tag{3.21}$$

Here $(\Phi_\tau)_* \rho$ is the push-forward measure (defined by $((\Phi_\tau)_* \rho)(\Omega) := \rho(\Phi_\tau^{-1}(\Omega))$).

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Theorem 3.2.5. *Let ρ be a minimizing measure and Φ_τ be a continuously differentiable symmetry of the universal measure. Then for any compact subset $\Omega \subset M$,*

$$\left. \frac{d}{d\tau} \int_{\Omega} d\rho(x) \int_{M \setminus \Omega} d\rho(y) \left(\mathcal{L}(\Phi_\tau(x), y) - \mathcal{L}(x, \Phi_\tau(y)) \right) \right|_{\tau=0} = 0.$$

Proof. We again let f be a bounded measurable function on M . Then, by symmetry in x and y ,

$$\iint_{M \times M} f(x) f(y) \left(\mathcal{L}(\Phi_\tau(x), y) - \mathcal{L}(x, \Phi_\tau(y)) \right) d\rho(x) d\rho(y) = 0.$$

We replace $f(y)$ by $1 - (1 - f(y))$ and multiply out. The double integrals which do not involve $f(y)$ can be simplified as follows,

$$\begin{aligned} \iint_{M \times M} f(x) \mathcal{L}(\Phi_\tau(x), y) d\rho(x) d\rho(y) &= \int_M f(x) \ell(\Phi_\tau(x)) d\rho(x) \\ \iint_{M \times M} f(x) \mathcal{L}(x, \Phi_\tau(y)) d\rho(x) d\rho(y) &= \iint_{\mathcal{F} \times \mathcal{F}} f(x) \mathcal{L}(x, \Phi_\tau(y)) d\rho(x) d\rho(y) \\ &= \iint_{\mathcal{F} \times \mathcal{F}} f(x) \mathcal{L}(x, y) d\rho(x) d((\Phi_\tau)_*\rho)(y) \stackrel{(\star)}{=} \iint_{\mathcal{F} \times \mathcal{F}} f(x) \mathcal{L}(x, y) d\rho(x) d\rho(y) \\ &= \iint_{M \times M} f(x) \mathcal{L}(x, y) d\rho(x) d\rho(y) = \int_M f(x) \ell(x) d\rho(x), \end{aligned}$$

where in (\star) we used the symmetry assumption (3.21). We thus obtain

$$\begin{aligned} 0 &= - \iint_{M \times M} f(x) (1 - f(y)) \left(\mathcal{L}(\Phi_\tau(x), y) - \mathcal{L}(x, \Phi_\tau(y)) \right) d\rho(x) d\rho(y) \\ &\quad + \int_M f(x) \left(\ell(\Phi_\tau(x)) - \ell(x) \right) d\rho(x). \end{aligned}$$

Choosing f as the characteristic function of Ω gives

$$\int_{\Omega} d\rho(x) \int_{M \setminus \Omega} d\rho(y) \left(\mathcal{L}(\Phi_\tau(x), y) - \mathcal{L}(x, \Phi_\tau(y)) \right) = \int_{\Omega} \left(\ell(\Phi_\tau(x)) - \ell(x) \right) d\rho(x).$$

Now the τ -derivative can be computed just as in the proof of Theorem 3.2.3. \square

3.2.3 Generalized Integrated Symmetries

We now combine the symmetries of the previous sections in the notion of “generalized integrated symmetries.” Our method is based on the following simple but useful identity.

Proposition 3.2.6. *Let Φ_τ be a variation of the form (3.14). Then*

$$\int_M d\rho(x) \int_\Omega d\rho(y) \left(\mathcal{L}(\Phi_\tau(x), y) - \mathcal{L}(x, y) \right) \quad (3.22)$$

$$= \int_\Omega \left(\ell(\Phi_\tau(x)) - \ell(x) \right) d\rho(x) \quad (3.23)$$

$$- \int_\Omega d\rho(x) \int_{M \setminus \Omega} d\rho(y) \left(\mathcal{L}(\Phi_\tau(x), y) - \mathcal{L}(x, \Phi_\tau(y)) \right). \quad (3.24)$$

Proof. We rewrite the integration domains as follows,

$$\begin{aligned} & \int_M d\rho(x) \int_\Omega d\rho(y) \left(\mathcal{L}(\Phi_\tau(x), y) - \mathcal{L}(x, y) \right) \\ &= \int_\Omega d\rho(x) \int_\Omega d\rho(y) \left(\mathcal{L}(\Phi_\tau(x), y) - \mathcal{L}(x, y) \right) \\ & \quad + \int_{M \setminus \Omega} d\rho(x) \int_\Omega d\rho(y) \left(\mathcal{L}(\Phi_\tau(x), y) - \mathcal{L}(x, y) \right) \\ &= \int_\Omega d\rho(x) \int_M d\rho(y) \left(\mathcal{L}(\Phi_\tau(x), y) - \mathcal{L}(x, y) \right) \end{aligned} \quad (3.25)$$

$$- \int_\Omega d\rho(x) \int_{M \setminus \Omega} d\rho(y) \left(\mathcal{L}(\Phi_\tau(x), y) - \mathcal{L}(x, y) \right) \quad (3.26)$$

$$+ \int_{M \setminus \Omega} d\rho(x) \int_\Omega d\rho(y) \left(\mathcal{L}(\Phi_\tau(x), y) - \mathcal{L}(x, y) \right). \quad (3.27)$$

In (3.25) we can carry out the y -integration using (3.7). In (3.27) we exchange the integrals and use that the Lagrangian is symmetric in its two arguments (3.5). This gives the result. \square

Note that the term (3.24) is a surface layer integral. The term (3.23), on the other hand, only involves ℓ , and therefore its first variation vanishes in view of the EL equations (3.8). We thus obtain a conservation law, provided that the term (3.22) vanishes. This motivates the following definition.

Definition 3.2.7. *A variation Φ_τ of the form (3.14) is a **generalized integrated symmetry** in the space-time region $\Omega \subset M$ if*

$$\int_M d\rho(x) \int_\Omega d\rho(y) \left(\mathcal{L}(\Phi_\tau(x), y) - \mathcal{L}(x, y) \right) = 0. \quad (3.28)$$

This notion of symmetry indeed generalizes our previous notions of symmetry (see Definitions 3.2.2 and 3.2.4) in the sense that symmetries of the Lagrangian and of the universal measure imply that (3.28) holds for first variations. Namely, if Φ_τ is a symmetry

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of the universal measure, we can use (3.21) to obtain

$$\begin{aligned} & \int_M d\rho(x) \int_\Omega d\rho(y) \left(\mathcal{L}(\Phi_\tau(x), y) - \mathcal{L}(x, y) \right) \\ &= \int_{\mathcal{F}} d((\Phi_\tau)_*\rho)(x) \int_\Omega d\rho(y) \mathcal{L}(x, y) - \int_{\mathcal{F}} d\rho(x) \int_\Omega d\rho(y) \mathcal{L}(x, y) = 0. \end{aligned} \quad (3.29)$$

Likewise, if Φ_τ is a symmetry of the Lagrangian, we can apply (3.16). This gives the identity

$$\begin{aligned} & \int_M d\rho(x) \int_\Omega d\rho(y) \left(\mathcal{L}(\Phi_\tau(x), y) - \mathcal{L}(x, y) \right) \\ &= \int_M d\rho(x) \int_\Omega d\rho(y) \left(\mathcal{L}(x, \Phi_{-\tau}(y)) - \mathcal{L}(x, y) \right) \\ &= \int_\Omega \left(\ell(\Phi_{-\tau}(y)) - \ell(y) \right) d\rho(y), \end{aligned} \quad (3.30)$$

whose first variation vanishes in view of (3.20).

Combining Definition 3.2.7 with Proposition 3.2.6 immediately gives the following result.

Theorem 3.2.8. *Let ρ be a minimizing measure and Φ_τ a continuously differentiable generalized integrated symmetry (see Definition 3.2.7). Then for any compact subset $\Omega \subset M$,*

$$\left. \frac{d}{d\tau} \int_\Omega d\rho(x) \int_{M \setminus \Omega} d\rho(y) \left(\mathcal{L}(\Phi_\tau(x), y) - \mathcal{L}(x, \Phi_\tau(y)) \right) \right|_{\tau=0} = 0.$$

In view of (3.29) and (3.30), the previous conservation laws of Theorems 3.2.3 and 3.2.5 are immediate corollaries of this theorem.

3.3 The Setting of Causal Fermion Systems

We now turn attention to the setting of causal fermion systems. After a short review of the mathematical framework and the Euler-Lagrange equations (Section 3.3.1), we prove Noether-like theorems (Section 3.3.2). The reader interested in a more detailed introduction to causal fermion systems is referred to Chapter 2 and to the introductory chapter in [Fin16b].

3.3.1 Basic Definitions and the Euler-Lagrange Equations

Definition 3.3.1. (causal fermion system) Given a separable complex Hilbert space \mathcal{H} with scalar product $\langle \cdot | \cdot \rangle_{\mathcal{H}}$ and a parameter $n \in \mathbb{N}$ (the “*spin dimension*”), we let $\mathcal{F} \subset L(\mathcal{H})$ be the set of all self-adjoint operators on \mathcal{H} of finite rank, which (counting multiplicities) have at most n positive and at most n negative eigenvalues. On \mathcal{F} we are given a positive measure ρ (defined on a σ -algebra of subsets of \mathcal{F}), the so-called *universal measure*. We refer to $(\mathcal{H}, \mathcal{F}, \rho)$ as a *causal fermion system*.

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We next introduce the causal action principle. For any $x, y \in \mathcal{F}$, the product xy is an operator of rank at most $2n$. We denote its non-trivial eigenvalues (counting algebraic multiplicities) by $\lambda_1^{xy}, \dots, \lambda_{2n}^{xy} \in \mathbb{C}$. We introduce the *spectral weight* $|\cdot|$ of an operator as the sum of the absolute values of its eigenvalues. In particular, the spectral weight of the operator products xy and $(xy)^2$ is defined by

$$|xy| = \sum_{i=1}^{2n} |\lambda_i^{xy}| \quad \text{and} \quad |(xy)^2| = \sum_{i=1}^{2n} |\lambda_i^{xy}|^2.$$

We introduce the Lagrangian and the action by

$$\text{Lagrangian:} \quad \mathcal{L}(x, y) = |(xy)^2| - \frac{1}{2n} |xy|^2 \quad (3.31)$$

$$\text{action:} \quad \mathcal{S}(\rho) = \iint_{\mathcal{F} \times \mathcal{F}} \mathcal{L}(x, y) d\rho(x) d\rho(y). \quad (3.32)$$

The *causal action principle* is to minimize \mathcal{S} by varying the universal measure under the following constraints:

$$\text{volume constraint:} \quad \rho(\mathcal{F}) = \text{const} > 0 \quad (3.33)$$

$$\text{trace constraint:} \quad \int_{\mathcal{F}} \text{tr}(x) d\rho(x) = \text{const} \neq 0 \quad (3.34)$$

$$\text{boundedness constraint:} \quad \mathcal{T}(\rho) := \iint_{\mathcal{F} \times \mathcal{F}} |xy|^2 d\rho(x) d\rho(y) \leq C, \quad (3.35)$$

where C is a given parameter (and tr denotes the trace of linear operators on \mathcal{H}).

3.3.1.1 The finite-dimensional setting

If \mathcal{H} is *finite-dimensional* and ρ has *finite total volume*, the existence of minimizers is proven in [Fin10a], and the corresponding EL equations are derived in [BF14]. We now recall a few of these results. Under the above assumptions, on \mathcal{F} one considers the topology induced by the operator norm

$$\|A\| := \sup \{ \|Au\|_{\mathcal{H}} \text{ with } \|u\|_{\mathcal{H}} = 1 \}. \quad (3.36)$$

In this topology, the Lagrangian as well as the integrands in (3.34) and (3.35) are continuous. We vary ρ within the class of bounded Borel measures of \mathcal{F} . The existence of minimizers of the action (3.32) under the constraints (3.33)–(3.35) is proven in [Fin10a, Theorem 2.1]. For our purposes, the resulting EL equations are most conveniently stated as follows (for a heuristic derivation see the introduction in [BF14]).

Theorem 3.3.2. *Assume that ρ is a minimizer of the causal action principle for C so large that*

$$C > \inf \{ \mathcal{T}(\mu) \mid \mu \text{ satisfies (3.33) and (3.34)} \}. \quad (3.37)$$

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Moreover, assume that one of the following two technical assumptions hold:

(i) The boundedness constraint is satisfied with a strict inequality,

$$\mathcal{T}(\rho) < C. \quad (3.38)$$

(ii) The minimizer is regular in the sense of [BF14, Definition 3.12].

Then for a suitable choice of Lagrange multipliers $\lambda, \kappa \in \mathbb{R}$, the measure ρ is supported on the intersection of the level sets

$$\Phi_1(x) = -4\mathcal{S}(\rho) \quad \text{and} \quad \Phi_2(x) = 2\mathcal{S}(\rho), \quad (3.39)$$

where

$$\Phi_1(x) := -\lambda \operatorname{tr}(x), \quad \Phi_2(x) := 2 \int_{\mathcal{F}} \mathcal{L}_\kappa(x, y) d\rho(y) \quad (3.40)$$

and

$$\mathcal{L}_\kappa(x, y) := \mathcal{L}(x, y) + \kappa |xy|^2. \quad (3.41)$$

Moreover, the function

$$\Phi(x) := \Phi_1 + \Phi_2$$

is minimal on the support of ρ , i.e.

$$\Phi|_{\operatorname{supp} \rho} = \inf_{\mathcal{F}} \Phi. \quad (3.42)$$

Proof. We first apply [BF14, Theorem 1.3] to the causal variational principle with trace constraint in the case $\mathcal{T}(\rho) < C$. This yields that ρ is supported on the intersection of the level sets (3.39). Moreover, this theorem implies that $\Phi|_{\operatorname{supp} \rho} = -2\mathcal{S}(\rho)$. The minimality (3.42) is proven in [BF14, Theorem 3.13], noting that the regularity condition of [BF14, Definition 3.12] is automatically satisfied if the trace constraint is considered and if (3.38) holds. \square

We remark for clarity that the inequality (3.37) can always be arranged by choosing C sufficiently large. The assumptions (i) or (ii) are needed in order for the Lagrange multiplier method to be applicable. The basic difficulty comes about because the set of positive Borel measures is not a vector space, but only a convex set. Moreover, one must make sure that the constraints describe locally a Banach submanifold. We refer the reader interested in the technical details to the paper [BF14]. In what follows, we take the assumptions (i) or (ii) for granted.

For the derivation of our conservation laws, we only need a weaker version of the EL equations (3.42). Namely, it suffices to assume that the function Φ is constant on the support of ρ ,

$$\Phi|_{\operatorname{supp} \rho} = \operatorname{const}, \quad (3.43)$$

and that the support of ρ is a *local* minimum in the sense that every $x \in \operatorname{supp} \rho$ has a

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neighborhood $U(x) \subset \mathcal{F}$ such that

$$\Phi(x) = \inf_{U(x)} \Phi. \quad (3.44)$$

We subsume (3.43) and (3.44) by saying that ρ is a *local minimizer* of the causal action principle. Working with local minimizers is also preferable because the regularized Dirac sea configurations to be considered in the examples of Sections 3.4.2 and 3.5.2 are known to satisfy (3.43) and (3.44) in the continuum limit, but they are not global minimizers of the causal action principle (for a detailed discussion of this point in the connection to microscopic mixing and second-quantized bosonic fields we refer to [Fin16b, §1.5.3]).

3.3.1.2 The infinite-dimensional setting

We next consider the case that \mathcal{H} is infinite-dimensional or the total volume $\rho(\mathcal{F})$ is infinite. First, a scaling argument shows that in the case $\rho(\mathcal{F}) = \infty$ and $\dim \mathcal{H} < \infty$, the action is infinite for all measures satisfying the constraints, so that the variational principle is not sensible. Similarly, if $\rho(\mathcal{F}) < \infty$ and $\dim \mathcal{H} = \infty$, the infimum of the action is zero, but this infimum is not attained (for details see [Fin16b, Exercise 1.3]). Therefore, the only interesting case is the *infinite-dimensional* setting when $\rho(\mathcal{F}) = \infty$ and $\dim \mathcal{H} = \infty$. In this setting, the causal action principle makes mathematical sense if the volume constraint (3.33) is implemented by demanding that the variations $(\rho(\tau))_{\tau \in (-\tau_{\max}, \tau_{\max})}$ should for all $\tau, \tau' \in (-\tau_{\max}, \tau_{\max})$ satisfy the conditions

$$|\rho(\tau) - \rho(\tau')|(\mathcal{F}) < \infty \quad \text{and} \quad (\rho(\tau) - \rho(\tau'))(\mathcal{F}) = 0$$

(where $|\cdot|$ denotes the total variation of a measure; see [Hal74, §28]). But the existence of minimizers has not yet been proven. Nevertheless, the EL equations are well-defined in the following sense:

Definition 3.3.3. *Let $(\rho, \mathcal{H}, \mathcal{F})$ be a causal fermion system (possibly with $\dim \mathcal{H} = \infty$ and $\rho(\mathcal{F}) = \infty$). The measure ρ is a **local minimizer** of the causal action principle if the integral in (3.40) is finite for all $x \in \mathcal{F}$ and if the EL equations (3.43) and (3.44) hold for a suitable parameter $\lambda \in \mathbb{R}$.*

Such local minimizers arise naturally when analyzing the continuum limit of causal fermion systems (see [Fin16b]). Also, the physical examples in Sections 3.4 and 3.5 will be formulated for local minimizers in the infinite-dimensional setting. Finally, the above notion of local minimizers is of relevance in view of future extensions of the existence theory to the infinite-dimensional setting.

Let ρ be a local minimizer of the causal action principle. We again define *space-time* by $M = \text{supp } \rho$; it is a closed but in general non-compact subset of $\mathcal{F} \subset L(\mathcal{H})$. We again define the function ℓ by

$$\ell(x) = \int_M \mathcal{L}_\kappa(x, y) d\rho(y) \quad (3.45)$$

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and for notational convenience set $\nu = \lambda/2$. By assumption, this function is well-defined and finite for all $x \in \mathcal{F}$. Moreover, the EL equations (3.43) and (3.44) imply that

$$\begin{aligned} \ell(x) - \nu \operatorname{tr}(x) & \text{ is constant on } M \\ \ell(x) - \nu \operatorname{tr}(x) &= \inf_{y \in U(x)} (\ell(y) - \nu \operatorname{tr}(y)) \quad \text{ for all } x \in M \end{aligned} \quad (3.46)$$

(where $U(x) \subset \mathcal{F}$ is again a neighborhood of x). However, the function ℓ need not be integrable. In particular, the action (3.32) may be infinite.

These EL equations imply analogs of the relations (3.39) and (3.40). Namely, evaluating the identity

$$\frac{d}{dt} (\ell(tx) - \nu \operatorname{tr}(tx)) \Big|_{t=1} = 0$$

and using that the Lagrangian (3.31) is homogeneous of degree two, one finds that on M ,

$$2\ell(x) - \nu \operatorname{tr}(x) = 0.$$

Combining this relation with (3.46), one concludes that on M , the two terms in (3.46) are separately constant, i.e.

$$\ell(x) = - \inf_{y \in \mathcal{F}} (\ell(y) - \nu \operatorname{tr}(y)) = \frac{\nu}{2} \operatorname{tr}(x) \quad \text{ for all } x \in M. \quad (3.47)$$

These identities are very useful because they show that on M , both summands in (3.46) are separately constant. Moreover, these relations make it possible to compute the Lagrange multiplier ν .

3.3.2 Noether-Like Theorems

Let $(\mathcal{H}, \mathcal{F}, \rho)$ be a causal fermion system, where ρ is a local minimizer of the causal action (see Definition 3.3.3). We do not want to assume that \mathcal{H} is finite-dimensional nor that the total volume of ρ is finite. But we shall assume that ρ is **locally finite** in the sense that $\rho(K) < \infty$ for every compact subset $K \subset \mathcal{F}$.

We again consider variations Φ_τ of M in \mathcal{F} described by a mapping Φ of the form (3.14),

$$\Phi : (-\tau_{\max}, \tau_{\max}) \times M \rightarrow \mathcal{F} \quad \text{ with } \quad \Phi(0, \cdot) = \mathbb{1}. \quad (3.48)$$

Similar to Definition 3.2.1, the regularity of the variation is defined by composing Φ with an operator mapping to the real numbers. However, we now compose both with ℓ and with the trace operation.

Definition 3.3.4. A variation Φ_τ of the form (3.48) is **continuous** if the compositions

$$\ell \circ \Phi, \operatorname{tr} \circ \Phi : (-\tau_{\max}, \tau_{\max}) \times M \rightarrow \mathbb{R}$$

are continuous. If in addition their partial derivative $\partial_\tau(\ell \circ \Phi)$ and $\partial_\tau(\operatorname{tr} \circ \Phi)$ exist and are continuous on $(-\tau_{\max}, \tau_{\max}) \times M \rightarrow \mathbb{R}$, then the variation is said to be **continuously differentiable**.

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We now generalize Proposition 3.2.6 to the setting of causal fermion systems.

Proposition 3.3.5. *Let Φ_τ be a continuous variation of the form (3.48). Then for any compact subset $\Omega \subset M$,*

$$\int_M d\rho(x) \int_\Omega d\rho(y) \left(\mathcal{L}_\kappa(\Phi_\tau(x), y) - \mathcal{L}_\kappa(x, y) \right) \quad (3.49)$$

$$= \int_\Omega \left(\ell(\Phi_\tau(x)) - \ell(x) \right) d\rho(x) \quad (3.50)$$

$$- \int_\Omega d\rho(x) \int_{M \setminus \Omega} d\rho(y) \left(\mathcal{L}_\kappa(\Phi_\tau(x), y) - \mathcal{L}_\kappa(x, \Phi_\tau(y)) \right). \quad (3.51)$$

Proof. The subtle point is that M is in general non-compact, so that some of the integrals may diverge. Therefore, we need to carefully consider the different integrals one after each other: From (3.47) we know that the functions ℓ and $\text{tr}(x)$ are both constant on M . Moreover, the functions $\ell \circ \Phi$ and $\text{tr} \circ \Phi$ are continuous on $(-\tau_{\max}, \tau_{\max}) \times M$. As a consequence, it follows that for any compact subset $\Omega \subset M$ and any $\delta < \tau_{\max}$, the restriction

$$\ell \circ \Phi|_{[-\delta, \delta] \times \Omega} : [-\delta, \delta] \times \Omega \rightarrow \mathbb{R}$$

is a bounded function. Using that the Lagrangian is non-negative, this implies that for any $\tau \in (-\delta, \delta)$, the double integrals of the form

$$\int_\Omega d\rho(x) \int_U d\rho(y) \mathcal{L}_\kappa(\Phi_\tau(x), y)$$

are well-defined and finite for any measurable subset $U \subset M$. Moreover, one may exchange the orders of integration using Tonelli's theorem (i.e. the version of Fubini's theorem for non-negative integrands). In particular, we conclude that the following integrals in (3.49) and (3.51) are well-defined and finite,

$$\int_M d\rho(x) \int_\Omega d\rho(y) \mathcal{L}_\kappa(x, y) \quad \text{and} \quad \int_\Omega d\rho(x) \int_{M \setminus \Omega} d\rho(y) \mathcal{L}_\kappa(\Phi_\tau(x), y).$$

For the integral in (3.50), we can argue similarly: We saw above that the functions $\ell \circ \Phi$ and $\text{tr} \circ \Phi$ are bounded on $\{0\} \times M$ and continuous on $(-\tau_{\max}, \tau_{\max}) \times M$. Therefore, they are bounded on $[-\delta, \delta] \times \Omega$, implying that the integral in (3.50) is well-defined and finite.

It remains to consider the two integrals

$$\int_M d\rho(x) \int_\Omega d\rho(y) \mathcal{L}_\kappa(\Phi_\tau(x), y) \quad \text{and} \quad \int_\Omega d\rho(x) \int_{M \setminus \Omega} d\rho(y) \mathcal{L}_\kappa(x, \Phi_\tau(y)). \quad (3.52)$$

These integrals could diverge. But since the integrand is non-negative, Tonelli's theorem nevertheless allows us to exchange the two integrals. Then the integrands of the two integrals coincide. The integration ranges coincide up to the compact set $\Omega \times \Omega$. Therefore,

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the first integral in (3.52) diverges if and only if the second integral diverges. If this is the case, the left and the right side of the equation (3.49)–(3.51) both take the value $+\infty$, so that the statement of the proposition holds. In the case that the integrals in (3.52) are both finite, we can repeat the computation in the proof of Proposition 3.2.6 and apply (3.45) to obtain the result. \square

Definition 3.3.6. *The variation Φ_τ is a **generalized integrated symmetry** in the space-time region $\Omega \subset M$ if the following two identities hold:*

$$\int_M d\rho(x) \int_\Omega d\rho(y) \left(\mathcal{L}_\kappa(\Phi_\tau(x), y) - \mathcal{L}_\kappa(x, y) \right) = 0 \quad (3.53)$$

$$\int_\Omega \left(\text{tr}(\Phi_\tau(x)) - \text{tr}(x) \right) d\rho(x) = 0. \quad (3.54)$$

Combining this definition with Proposition 3.3.5 and the EL equations (3.46) immediately gives the following result:

Theorem 3.3.7. *Let ρ be a local minimizer of the causal action (see Definition 3.3.3) and Φ_τ a continuously differentiable generalized integrated symmetry (see Definitions 3.3.4 and 3.3.6). Then for any compact subset $\Omega \subset M$, the following surface layer integral vanishes,*

$$\frac{d}{d\tau} \int_\Omega d\rho(x) \int_{M \setminus \Omega} d\rho(y) \left(\mathcal{L}_\kappa(\Phi_\tau(x), y) - \mathcal{L}_\kappa(x, \Phi_\tau(y)) \right) \Big|_{\tau=0} = 0. \quad (3.55)$$

In order to explain the necessity of the condition (3.54), we point out that, although the functions $\ell(x)$ and $\text{tr}(x)$ are both constant on M (see (3.47)), this does not imply that transversal derivatives of these functions vanish. Only for their specific linear combination in (3.47) the derivative vanishes on M . We also note that the condition for the trace (3.54), which did not appear in the compact setting, can always be satisfied by rescaling the variation according to

$$\Phi_\tau(x) \rightarrow \Phi_\tau(x) \frac{\text{tr}(x)}{\text{tr}(\Phi_\tau(x))}$$

(note that by continuity, the trace in the denominator is non-zero for sufficiently small τ). However, when doing so, the remaining condition (3.53) as well as the regularity conditions of Definition 3.3.4 might become more involved. This is the reason why we prefer to write two separate conditions (3.53) and (3.54).

The above results give rise to corollaries which extend Theorems 3.2.3 and 3.2.5 to the setting of causal fermion systems.

Definition 3.3.8. *A variation Φ_τ of the form (3.48) is a **symmetry of the Lagrangian** if*

$$\mathcal{L}_\kappa(x, \Phi_\tau(y)) = \mathcal{L}_\kappa(\Phi_{-\tau}(x), y) \quad \text{for all } \tau \in (-\tau_{\max}, \tau_{\max}) \text{ and all } x, y \in M. \quad (3.56)$$

It is a *symmetry of the universal measure* if

$$(\Phi_\tau)_*\rho = \rho \quad \text{for all } \tau \in (-\tau_{\max}, \tau_{\max}).$$

Moreover, it *preserves the trace* if

$$\text{tr}(\Phi_\tau(x)) = \text{tr}(x) \quad \text{for all } \tau \in (-\tau_{\max}, \tau_{\max}) \text{ and all } x \in M.$$

Corollary 3.3.9. *Let ρ be a local minimizer of the causal action (see Definition 3.3.3) and Φ_τ a continuously differentiable variation. Assume that Φ_τ is a symmetry of the Lagrangian and preserves the trace. Then for any compact subset $\Omega \subset M$, the conservation law (3.55) holds.*

Corollary 3.3.10. *Let ρ be a local minimizer of the causal action (see Definition 3.3.3) and Φ_τ a continuously differentiable variation. Assume that Φ_τ is a symmetry of the universal measure and preserves the trace. Then for any compact subset $\Omega \subset M$, the conservation law (3.55) holds.*

These corollaries follow immediately by calculations similar to (3.29) and (3.30).

3.4 Example: Current Conservation

This section is devoted to the important example of current conservation, also referred to as charge conservation. For Dirac particles, the electric charge is (up to a multiplicative constant) given as the integral over the probability density. Therefore, charge conservation also corresponds to the conservation of the probability integral in quantum mechanics. In the context of the classical Noether theorem, charge conservation is a consequence of an internal symmetry of the system, which can be described by a phase transformation (3.4) of the wave function and is often referred to as global gauge symmetry. As we shall see in Section 3.4.1, causal fermion systems also have such an internal symmetry, giving rise to a general class of conservation laws (see Theorem 3.4.2). In Section 3.4.2, these conservation laws are evaluated for Dirac spinors in Minkowski space, giving a correspondence to the conservation of the Dirac current (see Theorem 3.4.3 and Corollary 3.4.4). In Section 3.4.3, we conclude with a few clarifying remarks.

3.4.1 A General Conservation Law

Let \mathcal{A} be a bounded symmetric operator on \mathcal{H} and

$$\mathcal{U}_\tau := \exp(i\tau\mathcal{A}) \tag{3.57}$$

be the corresponding one-parameter family of unitary transformations. We introduce the mapping

$$\Phi : \mathbb{R} \times \mathcal{F} \rightarrow \mathcal{F}, \quad \Phi(\tau, x) = \mathcal{U}_\tau x \mathcal{U}_\tau^{-1}. \tag{3.58}$$

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Restricting this mapping to $(-\tau_{\max}, \tau_{\max}) \times M$, we obtain a variation $(\Phi_\tau)_{\tau \in (-\tau_{\max}, \tau_{\max})}$ of the form (3.48).

Lemma 3.4.1. *The variation Φ_τ given by (3.58) is a symmetry of the Lagrangian and preserves the trace (see Definition 3.3.8).*

Proof. Since $\Phi_\tau(x)$ is unitarily equivalent to x , they obviously have the same trace. In order to prove (3.56), we first recall that the Lagrangian $\mathcal{L}_\kappa(x, y)$ is defined in terms of the spectrum of the operator product xy (see (3.31)). The calculation

$$x \Phi_\tau(y) = x \mathcal{U}_\tau y \mathcal{U}_\tau^{-1} = \mathcal{U} (\mathcal{U}_\tau^{-1} x \mathcal{U}_\tau y) \mathcal{U}_\tau^{-1} = \mathcal{U} (\Phi_{-\tau}(x) y) \mathcal{U}_\tau^{-1}$$

shows that the operators $x \Phi_\tau(y)$ and $\Phi_{-\tau}(x) y$ are unitarily equivalent and therefore isospectral. This concludes the proof. \square

It remains to verify whether the variation Φ_τ is continuously differentiable in the sense of Definition 3.3.4. For the trace, this is obvious because Φ_τ leaves the trace invariant, so that $\text{tr} \circ \Phi_\tau(\tau, x) = \text{tr}(x)$, which clearly depends continuously on x (in the topology induced by the sup-norm (3.36)). For $\ell \circ \phi$, we cannot in general expect differentiability because the Lagrangian \mathcal{L}_κ is only Lipschitz continuous in general. Therefore, we must include the differentiability of $\ell \circ \phi$ as an assumption in the following theorem.

Theorem 3.4.2. *Given a bounded symmetric operator \mathcal{A} on \mathcal{H} , we let Φ_τ be the variation (3.58). Assume that the mapping $\ell \circ \Phi : (-\tau_{\max}, \tau_{\max}) \times M \rightarrow \mathbb{R}$ is continuously differentiable in the sense that it is continuous and that $\partial_\tau(\ell \circ \Phi)$ exists and is also continuous on $(-\tau_{\max}, \tau_{\max}) \times M$. Then for any compact subset $\Omega \subset M$, the conservation law (3.55) holds.*

3.4.2 Correspondence to Dirac Current Conservation

The aim of this section is to relate the conservation law of Theorem 3.4.2 to the usual current conservation in relativistic quantum mechanics in Minkowski space.

To this end, we consider causal fermion systems $(\mathcal{F}, \mathcal{H}, \rho^\varepsilon)$ describing the regularized Dirac sea vacuum in Minkowski space $(\mathcal{M}, \langle \cdot, \cdot \rangle)$. We briefly recall the construction (for the necessary preliminaries see [Fin16a, Section 2], [Fin16b], [FG12, Section 4] or the introduction in Chapter 2. As in [Fin16b, Chapter 3] we consider three generations of Dirac particles of masses m_1, m_2 and m_3 (corresponding to the three generations of elementary particles in the standard model; three generations are necessary in order to obtain well-posed equations in the continuum limit). Denoting the generations by an index β , we consider the Dirac equations

$$(i\partial - m_\beta) \psi_\beta = 0 \quad (\beta = 1, 2, 3). \quad (3.59)$$

On solutions $\psi = (\psi_\beta)_{\beta=1,2,3}$, we consider the scalar product

$$(\psi|\phi) := 2\pi \sum_{\beta=1}^3 \int_{\mathbb{R}^3} (\bar{\psi}_\beta \gamma^0 \phi_\beta)(t, \vec{x}) d^3x.$$

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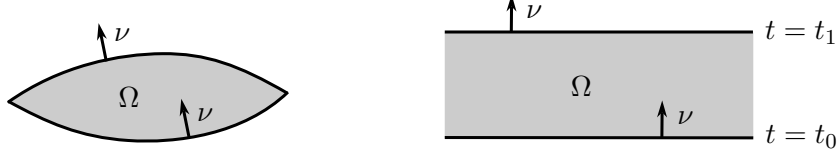


Figure 3.2: Choice of the space-time region $\Omega \subset \mathcal{M}$.

The Dirac equation has solutions on the upper and lower mass shell, which have positive respectively negative energy. In order to avoid potential confusion with other notions of energy, we here prefer the notion of solutions of positive and negative *frequency*. We choose \mathcal{H} as the subspace spanned by all solutions of negative frequency, together with the scalar product $\langle \cdot | \cdot \rangle_{\mathcal{H}} := (\cdot | \cdot)_{\mathcal{H} \times \mathcal{H}}$. We now introduce an ultraviolet regularization (for details see [Fin16a, Section 2]) and denote the regularized quantities by a superscript ε . Now the local correlation operators are defined by

$$\langle \psi^\varepsilon | F^\varepsilon(x) \phi^\varepsilon \rangle_{\mathcal{H}} = - \sum_{\alpha, \beta=1}^3 \overline{\psi_\alpha^\varepsilon(x)} \phi_\beta^\varepsilon(x) \quad \text{for all } \psi, \phi \in \mathcal{H}.$$

Next, the universal measure is defined as the push-forward of the Lebesgue measure $d\mu = d^4x$,

$$\rho^\varepsilon := (F_\varepsilon)_*(\mu).$$

Then $(\mathcal{H}, \mathcal{F}, \rho^\varepsilon)$ is a causal fermion system of spin dimension two. As shown in [Fin16b, Chapter 1], the kernel of the fermionic projector $P(x, y)$ converges as $\varepsilon \searrow 0$ to the distribution

$$P(x, y) = \sum_{\beta=1}^3 \int \frac{d^4k}{(2\pi)^4} (\not{k} + m_\beta) \delta(k^2 - m_\beta^2) e^{-ik(x-y)} \quad (3.60)$$

(this configuration is also referred to as three generations in a single sector; see [Fin16b, Chapter 3]). We remark that our ansatz can be generalized by introducing so-called weight factors (see [Fin08] and Remark 3.4.12 below).

We want to apply Theorem 3.4.2. Since in this theorem, the set Ω must be compact, we choose it as a lens-shaped region whose boundary is composed of two space-like hypersurfaces (see the left of Figure 3.2). Considering a sequence of compact sets Ω_n which exhaust the region Ω between two Cauchy surfaces at times $t = t_0$ and $t = t_1$, the surface layer integral (3.55) reduces to the difference of surface layers integrals at times $t \approx t_0$ and $t \approx t_1$. The detailed analysis (which will be carried out below) gives the following result:

Theorem 3.4.3. (Current conservation) *Let $(\mathcal{H}, \mathcal{F}, \rho^\varepsilon)$ be local minimizers of the causal action which describe the Minkowski vacuum (3.60). Considering the limiting procedure explained in Figure 3.2 and taking the continuum limit, the conservation laws of Theorem 3.4.2 go over to a linear combination of the probability integrals in every generation. More precisely, there are non-negative constants c_β such that for all $u \in \mathcal{H}$*

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for which ψ^u is a negative-frequency solution of the Dirac equation, the surface layer integral (3.55) goes over the equation

$$\sum_{\beta=1}^3 m_{\beta} c_{\beta} \int_{t=t_0} \langle \psi_{\beta}^u(x) | \gamma^0 \psi_{\beta}^u(x) \rangle d^3x = \sum_{\beta=1}^3 m_{\beta} c_{\beta} \int_{t=t_1} \langle \psi_{\beta}^u(x) | \gamma^0 \psi_{\beta}^u(x) \rangle d^3x. \quad (3.61)$$

The constants c_{β} depend on properties of the distribution \hat{Q} in the continuum limit, as will be specified in Definition 3.4.6 and (3.83) below.

Before coming to the proof, we explain the statement and significance of this theorem. We first note that the restriction to negative-frequency solutions is needed because the description of positive-frequency solutions involves the so-called mechanism of microscopic mixing which for brevity we cannot address in this paper (see however the remarks in Section 3.7 below). Next, we point out that the theorem implies the statement that the function $\ell \circ \Phi$ in Theorem 3.4.2 is continuously differentiable in the continuum limit. However, this does not necessarily mean that this differentiability statement holds for any local minimizer $(\mathcal{H}, \mathcal{F}, \rho^{\varepsilon})$ with regularization. This rather delicate technical point will be discussed in Remark 3.4.11 below.

Considering Cauchy hyperplanes in (3.61) is indeed no restriction because the theorem can be extended immediately to general Cauchy surfaces:

Corollary 3.4.4. (Current conservation on Cauchy surfaces) *Let $\mathcal{N}_0, \mathcal{N}_1$ be two Cauchy surfaces in Minkowski space, where \mathcal{N}_1 lies to the future of \mathcal{N}_0 . Then, under the assumptions of Theorem 3.4.3, the conservation law of Theorem 3.4.2 goes over to the conservation law for the current integrals*

$$\sum_{\beta=1}^3 m_{\beta} c_{\beta} \int_{\mathcal{N}_0} \langle \psi_{\beta}^u | \psi \psi_{\beta}^u \rangle d\mu_{\mathcal{N}_0} = \sum_{\beta=1}^3 m_{\beta} c_{\beta} \int_{\mathcal{N}_1} \langle \psi_{\beta}^u | \psi \psi_{\beta}^u \rangle d\mu_{\mathcal{N}_1}, \quad (3.62)$$

where ν denotes the future-directed normal.

Proof. We choose Ω as the space-time region between the two Cauchy surfaces. Using that the integrand in (3.55) is anti-symmetric in its arguments x and y , the integration range can be rewritten as

$$\begin{aligned} & \int_{\Omega} d\rho(x) \int_{M \setminus \Omega} d\rho(y) \left(\mathcal{L}_{\kappa}(\Phi_{\tau}(x), y) - \mathcal{L}_{\kappa}(x, \Phi_{\tau}(y)) \right) \\ &= \int_{J^{\wedge}(\mathcal{N}_1)} d\rho(x) \int_{J^{\vee}(\mathcal{N}_1)} d\rho(y) \left(\mathcal{L}_{\kappa}(\Phi_{\tau}(x), y) - \mathcal{L}_{\kappa}(x, \Phi_{\tau}(y)) \right) \\ & \quad - \int_{J^{\wedge}(\mathcal{N}_0)} d\rho(x) \int_{J^{\vee}(\mathcal{N}_0)} d\rho(y) \left(\mathcal{L}_{\kappa}(\Phi_{\tau}(x), y) - \mathcal{L}_{\kappa}(x, \Phi_{\tau}(y)) \right), \end{aligned} \quad (3.63)$$

where J^{\wedge} and J^{\vee} denote the causal past and causal future, respectively. For ease in notation, we refer to the integrals in (3.63) as a *surface layer integral over \mathcal{N}_1* . Thus the

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surface layer integral in (3.55) is the difference of two surface layer integrals over the Cauchy surfaces \mathcal{N}_0 and \mathcal{N}_1 .

In order to compute for example the surface layer integral over \mathcal{N}_0 , one chooses Ω as the region between the Cauchy surface \mathcal{N}_0 and the Cauchy surface $t = t_0$ (for sufficiently small t_0 ; in case that these Cauchy surfaces intersect for every t_0 , one modifies \mathcal{N}_0 near the asymptotic end without affecting our results). Applying the conservation law of Theorem 3.4.2 to this new region Ω , one concludes that the surface layer integral over \mathcal{N}_0 coincides with the surface layer integral at time $t \approx t_0$. The latter surface layer integral, on the other hand, was computed in Theorem 3.4.3 to go over to the sum of the probability integrals in (3.61). Finally, the usual current conservation for the Dirac dynamics shows that the integrals in (3.61) coincide with the surface integral over \mathcal{N}_0 in (3.62). This concludes the proof. \square

Using similar arguments, Theorem 3.4.3 can also be extended to interacting systems (see Remark 3.4.13 below).

The remainder of this section is devoted to the proof of Theorem 3.4.3. We first rewrite the causal action principle in terms of the kernel of the fermionic projector (for details see [Fin16b, §1.1]). The kernel of the fermionic projector $P(x, y)$ is defined by

$$P(x, y) = \pi_x y|_{S_y} : S_y \rightarrow S_x . \quad (3.64)$$

The *closed chain* is defined as the product

$$A_{xy} = P(x, y) P(y, x) : S_x \rightarrow S_x .$$

The nontrivial eigenvalues $\lambda_1^{xy}, \dots, \lambda^{xy}$ of the operator xy coincide with the eigenvalues of the closed chain. Moreover, it is useful to express $P(x, y)$ in terms of the wave evaluation operator defined by

$$\Psi(x) : \mathcal{H} \rightarrow S_x , \quad u \mapsto \psi^u(x) = \pi_x u . \quad (3.65)$$

Namely,

$$x = -\Psi(x)^* \Psi(x) \quad \text{and} \quad P(x, y) = -\Psi(x) \Psi(y)^* .$$

Our task is to compute the term $\mathcal{L}_\kappa(\Phi_\tau(x), y)$ in (3.55) for $x, y \in M$. The detailed computations in [Fin16b, §3.6.1] show that the fermionic projector of the Minkowski vacuum satisfies the EL equations in the continuum limit for $\kappa = 0$ (in our setting, this result means that the measures ρ^ε are local minimizers in the sense of Definition 3.3.3 in the limiting case $\varepsilon \searrow 0$). Therefore, we may set κ to zero. Thus our task is to compute the term $\mathcal{L}(\Phi_\tau(x), y)$. In preparation, we compute $P(\Phi_\tau(x), y)$. To this end, we first note that

$$\begin{aligned} \Phi_\tau(x) y &= \mathcal{U}_\tau x \mathcal{U}_\tau^{-1} y = \mathcal{U}_\tau \Psi(x)^* \Psi(x) \mathcal{U}_\tau^{-1} \Psi(y)^* \Psi(y) \\ &\simeq \Psi(x) \mathcal{U}_\tau^{-1} \Psi(y)^* \Psi(y) \mathcal{U}_\tau \Psi(x)^* , \end{aligned} \quad (3.66)$$

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where in the last line we cyclically commuted the operators and \simeq means that the operators are isospectral (up to irrelevant zeros in the spectrum). Therefore, introducing the notations

$$\Psi_\tau(x) = \Psi(x) \mathcal{U}_\tau^{-1} : \mathcal{H} \rightarrow S_x \quad (3.67)$$

$$P(\Phi_\tau(x), y) = -\Psi_\tau(x) \Psi(y)^*, \quad P(y, \Phi_\tau(x)) = -\Psi(y) \Psi_\tau(x)^* \quad (3.68)$$

one sees that the operator product $\Phi_\tau(x) y$ is isospectral to the modified closed chain

$$P(\Phi_\tau(x), y) P(y, \Phi_\tau(x)) . \quad (3.69)$$

Considering the Lagrangian as a function of this modified closed chain, the variation is described in a form suitable for computations.

For clarity, we explain in which sense the kernel of the fermionic projector as given by (3.68) agrees with the abstract definition (3.64),

$$P(\Phi_\tau(x), y) = \pi_{\Phi_\tau(x)} y . \quad (3.70)$$

It is a subtle point that the point $\Phi_\tau(x) \in \mathcal{F}$ depends on τ , so that space-time itself changes. However, when identifying the spin space $S_{\Phi_\tau(x)}$ with a corresponding spinor space in Minkowski space, the base point $x \in \mathcal{M}$ should be kept fixed. Therefore, the spin space $S_{\Phi_\tau(x)}$ is to be identified with the spinor space $S_x \mathcal{M}$. For each τ , this can be accomplished as explained above. This identification made, the kernel (3.68) indeed agrees with (3.70). The reason why we do not give the details of this construction is that the computation (3.66) already shows that the Lagrangian can be computed with the closed chain (3.69), and this is all we need for what follows.

We now choose $\mathcal{A} = \pi_{\langle u \rangle}$ as the projection on the one-dimensional subspace generated by a vector $u \in \mathcal{H}$ and let $\pi_{\langle u \rangle^\perp}$ be the projection on the orthogonal complement of u . Then

$$\begin{aligned} \Psi_\tau(x) &= \Psi(x) (\pi_{\langle u \rangle^\perp} + e^{-i\tau} \pi_{\langle u \rangle}) \\ P(\Phi_\tau(x), y) &= -\Psi(x) (\pi_{\langle u \rangle^\perp} + e^{-i\tau} \pi_{\langle u \rangle}) \Psi(y)^* \\ &= P(x, y) + (1 - e^{-i\tau}) \Psi(x) \pi_{\langle u \rangle} \Psi(y)^* . \end{aligned}$$

Normalizing u such that $\langle u|u \rangle_{\mathcal{H}} = 1$, the last equation can be written in the form that for any $\chi \in S_y$,

$$P(\Phi_\tau(x), y) \chi = P(x, y) \chi + (1 - e^{-i\tau}) \psi^u(x) \prec \psi^u(y) | \chi \succ_y .$$

We now compute the first order variation.

$$\begin{aligned} \frac{d}{d\tau} P(\Phi_\tau(x), y) \Big|_{\tau=0} \chi &= i\psi^u(x) \prec \psi^u(y) | \chi \succ_y =: \delta P(x, y) \chi \\ \frac{d}{d\tau} P(y, \Phi_\tau(x)) \Big|_{\tau=0} &= (\delta P(x, y))^* \end{aligned} \quad (3.71)$$

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The variation of the Lagrangian can be written as (cf. [Fin07a, Section 5.2] or [Fin16b, Section 1.4])

$$\begin{aligned}\delta\mathcal{L}(x, y) &:= \frac{d}{d\tau} \mathcal{L}(\Phi_\tau(x), y) \Big|_{\tau=0} \\ &= \text{Tr}_{S_y} (Q(y, x) \delta P(x, y)) + \text{Tr}_{S_x} (Q(x, y) \delta P(x, y)^*) \\ &= i \prec \psi^u(y) | Q(y, x) \psi^u(x) \succ_y - i \prec \psi^u(x) | Q(x, y) \psi^u(y) \succ_x ,\end{aligned}\tag{3.72}$$

where in the last line we used (3.71), and $Q(x, y)$ is a distributional kernel to be specified below. Using that the kernel $Q(x, y)$ is symmetric in the sense that

$$Q(x, y)^* = Q(y, x) ,$$

we can write the variation of the Lagrangian in the compact form

$$\delta\mathcal{L}(x, y) = -2 \text{Im} (\prec \psi^u(y) | Q(y, x) \psi^u(x) \succ_y) .$$

Using this identity, the surface layer integral in (3.55) can be written as

$$\int_{\Omega} d^4x \int_{\mathcal{M} \setminus \Omega} d^4y \text{Im} (\prec \psi^u(y) | Q(y, x) \psi^u(x) \succ_y) = 0 .$$

Taking the liming procedure as shown in Figure 3.2, it suffices to consider a surface layer integral at a fixed time t_0 , which for convenience we choose equal to zero. Thus our task is to compute the double integral

$$J := \int_{t \geq 0} d^4x \int_{t < 0} d^4y \text{Im} (\prec \psi^u(y) | Q(y, x) \psi^u(x) \succ) .\tag{3.73}$$

Here we omitted the subscript y at the spin scalar product because in Minkowski space all spinor spaces can be naturally identified.

In order to explain our method for computing the integrals in (3.73), we first state a simple lemma where integrals of this type are computed. As will be explained below, this lemma cannot be applied to our problem for technical reasons, but it nevertheless clarifies the structure of our results.

Lemma 3.4.5. *Let $f : \mathcal{M} \times \mathcal{M} \rightarrow \mathbb{R}$ be an integrable function with the following properties:*

- (a) *f is anti-symmetric, i.e. $f(x, y) = -f(y, x)$.*
- (b) *f is homogeneous in the sense that it depends only on the difference vector $y - x$.*
- (c) *The following integral is finite,*

$$\int_{\mathcal{M}} |x^0 f(x, 0)| d^4x < \infty .\tag{3.74}$$

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Then

$$\int_{-\infty}^0 dt \int_0^{\infty} dt' \int_{\mathbb{R}^3} d^3 y f((t, \vec{x}), (t', \vec{y})) = \frac{i}{2} \frac{\partial}{\partial k^0} \hat{f}(k) \Big|_{k=0}, \quad (3.75)$$

where \hat{f} is the Fourier transform, i.e.

$$f(x, y) = \int \frac{d^4 k}{(2\pi)^4} \hat{f}(k) e^{-ik(x-y)}. \quad (3.76)$$

Proof. Substituting (3.76) into the left side of (3.75), we can carry out the spatial integral to obtain

$$\int_{-\infty}^0 dt \int_0^{\infty} dt' \int_{\mathbb{R}^3} d^3 y f((t, \vec{x}), (t', \vec{y})) = \int_{-\infty}^0 dt \int_0^{\infty} dt' g(t - t'), \quad (3.77)$$

where

$$g(\tau) = \int_{\mathbb{R}^3} f((\tau, \vec{x}), (0, \vec{y})) d^3 y = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \hat{f}((\omega, \vec{0})) e^{-i\omega\tau}. \quad (3.78)$$

We now transform variables in the inner integral in (3.77),

$$\int_0^{\infty} g(t - t') dt' = \int_{-\infty}^t g(\tau) d\tau = \int_{-\infty}^0 g(\tau) \Theta(t - \tau) d\tau.$$

Using (3.74) and (3.78), we know that

$$\iint_{\mathbb{R}^- \times \mathbb{R}^-} |g(\tau) \Theta(t - \tau)| dt d\tau = \int_{-\infty}^0 |\tau g(\tau)| d\tau \leq \int_{\mathcal{M}} |x^0 f(x, 0)| d^4 x < \infty.$$

Hence in (3.77) we may switch the order of integration according to Fubini's theorem to obtain

$$\begin{aligned} \int_{-\infty}^0 dt \int_0^{\infty} dt' g(t - t') &= \int_{-\infty}^0 d\tau g(\tau) \int_{-\infty}^0 dt \Theta(t - \tau) \\ &= \int_{-\infty}^0 d\tau g(\tau) \int_{\tau}^0 dt = - \int_{-\infty}^0 d\tau \tau g(\tau) = -\frac{1}{2} \int_{-\infty}^{\infty} d\tau \tau g(\tau), \end{aligned}$$

where in the last step we used the anti-symmetry of g . Now we insert (3.78) and apply Plancherel's theorem,

$$\begin{aligned} \int_{-\infty}^{\infty} dt \int_0^{\infty} dt' g(t - t') &= -\frac{i}{2} \int_{-\infty}^{\infty} d\tau \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \hat{f}((\omega, \vec{0})) \frac{\partial}{\partial \omega} e^{-i\omega\tau} \\ &= \frac{i}{2} \int_{-\infty}^{\infty} d\tau \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \left(\frac{\partial}{\partial \omega} \hat{f}((\omega, \vec{0})) \right) e^{-i\omega\tau} = \frac{i}{2} \frac{\partial}{\partial \omega} \hat{f}((\omega, \vec{0})) \Big|_{\omega=0}. \end{aligned}$$

This concludes the proof. \square

In order to apply this lemma to our problem, we would have to show that the integrand

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in (3.73) satisfies the condition (3.74). As we shall now explain, this condition will indeed *not* be satisfied, making it necessary to modify the method.

Let us specify the kernel $Q(x, y)$. To this end, we make use of the fact that the fermionic projector of the vacuum should correspond to a stable minimizer of the causal action. This is made mathematically precise in the so-called state stability analysis carried out in [Fin07a, Section 5.6], [FH09] and [Fin08]. The detailed analysis of the continuum limit in [Fin16b, Chapter 3] shows that in order to obtain well-defined field equations in the continuum limit, the number of generations must be equal to three. Therefore, we now consider an unregularized fermionic projector of the vacuum involving a sum of three Dirac seas (3.60). The corresponding kernel $Q(x, y)$ obtained in the continuum limit depends only on the difference vector $y - x$ and can thus be written as the Fourier transform of a distribution $\hat{Q}(k)$,

$$Q(x, y) = \int \frac{d^4 k}{(2\pi)^4} \hat{Q}(k) e^{-ik(x-y)}.$$

The state stability analysis in [Fin07a, Section 5.6] implies that the Fourier transform \hat{Q} has the form as specified in the next definition (cf. [Fin07a, Definition 5.6.2]).

Definition 3.4.6. *The fermionic projector of the vacuum (3.60) is called **state stable** if the corresponding operator $\hat{Q}(k)$ is well-defined inside the lower mass cone*

$$\mathcal{C}^\wedge := \{k \in \mathbb{R}^4 \mid k^i k_i > 0 \text{ and } k^0 < 0\}$$

and can be written as

$$\hat{Q}(k) = a \frac{\not{k}}{|k|} + b \tag{3.79}$$

with continuous real functions a and b on \mathcal{C}^\wedge having the following properties:

(i) a and b are Lorentz invariant,

$$a = a(k^2), \quad b = b(k^2).$$

(ii) a is non-negative.

(iii) The function $a + b$ is minimal on the mass shells,

$$(a + b)(m_\beta^2) = \inf_{q \in \mathcal{C}^\wedge} (a + b)(q^2) \quad \text{for } \beta = 1, 2, 3. \tag{3.80}$$

We point out that, according to this definition, the function $\hat{Q}(k)$ does *not* need to be *smooth*, but only continuous. In particular, Lemma 3.4.5 cannot be applied, because the derivative in (3.75) is ill-defined. If $\hat{Q}(k)$ were smooth, its Fourier transform $Q(x, y)$ would decay rapidly as $(y - x)^2 \rightarrow \pm\infty$. In this case, $Q(x, y)$ would be of short range as explained in Section 3.1.3, except that (3.10) would have to be replaced by the statement that $\mathcal{L}(x, y)$ is very small if $|(y - x)^2| > \delta$ (and $\mathcal{L}(x, y)$ could indeed be made arbitrarily

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small by increasing δ). The fact that $\hat{Q}(k)$ does not need to be differentiable implies that $Q(x, y)$ does not need to decay rapidly, also implying that the condition (3.74) may be violated.

In fact, this non-smoothness in momentum space will be of importance in the following computation. Moreover, our results will depend only on the behavior $\hat{Q}(k)$ in a neighborhood of the mass shells $k^2 = m_\beta^2$. Therefore, the crucial role will be played by the regularity of \hat{Q} on the mass shells. In order to keep the setting as simple as possible, we shall assume that the functions a and b in (3.79) are *semi-differentiable* on the mass shells, meaning that the left and right derivatives exist. For the resulting semi-derivatives of \hat{Q} we use the notation

$$\begin{aligned}\partial_\omega^+ \hat{Q}(-\omega_{\beta, \vec{k}}, \vec{k}) &= \lim_{h \searrow 0} \frac{1}{h} \left(\hat{Q}(-\omega_{\beta, \vec{k}} + h, \vec{k}) - \hat{Q}(-\omega_{\beta, \vec{k}}, \vec{k}) \right) \\ \partial_\omega^- \hat{Q}(-\omega_{\beta, \vec{k}}, \vec{k}) &= \lim_{h \nearrow 0} \frac{1}{h} \left(\hat{Q}(-\omega_{\beta, \vec{k}} + h, \vec{k}) - \hat{Q}(-\omega_{\beta, \vec{k}}, \vec{k}) \right),\end{aligned}\tag{3.81}$$

where $\omega_{\beta, \vec{k}}$ is given by the dispersion relation

$$\omega_{\beta, \vec{k}} = \sqrt{m_\beta^2 + |\vec{k}|^2}.\tag{3.82}$$

The parameters c_β in Theorem 3.4.3 are given by

$$c_\beta := \partial_\omega^+ a(m_\beta^2) + \partial_\omega^+ b(m_\beta^2) + \partial_\omega^- a(m_\beta^2) + \partial_\omega^- b(m_\beta^2)\tag{3.83}$$

As explained above, even though the function $a + b$ is minimal at m_β^2 , it is in general not differentiable at this value. But the minimality implies that $c_\beta \geq 0$.

The discontinuity of the derivatives of \hat{Q} on the mass shells implies that $Q(x, y)$ will *not* decay rapidly as $(y - x)^2 \rightarrow \pm\infty$. Instead, we obtain contributions which decay only polynomially and oscillate on the *Compton scale* (this oscillatory behavior comes about similar as explained for the Fourier transforms of the mass shells in detail in [Fin16b, §1.2.5]). Due to these oscillations on the Compton scale, the integrals in (3.73) are indeed well-defined, and the dominant contribution to the integrals will come from a layer of width $\sim m^{-1}$ around the hyperplane $\{t = 0\}$. Therefore, although $\mathcal{L}(x, y)$ does not decay rapidly, the concept of the surface layer integral as introduced in Section 3.1.3 remains valid, and the parameter δ shown in Figure 3.1 can be identified with the Compton scale $\sim m_\alpha^{-1}$ of the Dirac particles. Thus the width of the surface layer is a small but macroscopic length scale. In particular, the surface layer integrals cannot be identified with or considered as a generalization of the surface integrals of the classical Noether theorem. However, in most situations of interest, when the surface is almost flat on the Compton scale, the surface layer integral can be well-approximated by a corresponding surface integral. Theorem 3.4.3 shows that in the limiting case that the surface is a hyperplane, the surface layer integral indeed goes over to a surface integral.

The just-mentioned oscillatory behavior of the integrand in (3.73) implies that the integrals will in general not exist in the Lebesgue sense. But they do exist in the sense of

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an improper Riemann integral. For computational purposes, this is implemented most conveniently by inserting convergence-generating factors. We begin with the simplest possible choice of a convergence-generating factor $e^{-\eta|t|}$. Thus instead of (3.73) we consider the integral

$$J = \lim_{\eta \searrow 0} \int_{t \geq 0} d^4x \int_{t < 0} d^4y e^{-\eta x^0 + \eta y^0} \operatorname{Im} (\prec \psi^u(y) | Q(y, x) \psi^u(x) \succ). \quad (3.84)$$

We now introduce a convenient representation for $\hat{\psi}^u(k)$. Since the wave function ψ^u is a linear combination of solutions of the Dirac equation corresponding to the masses m_β (with $\beta = 1, 2, 3$), its Fourier transform is supported on the mass shells $k^2 = m_\beta^2$. Moreover, since in the Dirac sea vacuum all physical wave functions have negative frequency, we can write $\hat{\psi}^u(k) = (\hat{\psi}_\beta^u(k))_{\beta=1,2,3}$ as

$$\hat{\psi}_\beta^u(k) = 2\pi \chi_\beta(\vec{k}) \delta(k^0 + \omega_{\beta, \vec{k}}) \quad (3.85)$$

(with $\omega_{\beta, \vec{k}}$ as in (3.82)). The Dirac equations (3.59) reduce to the algebraic equations

$$(k_\beta - m_\beta) \chi_\beta(\vec{k}) = 0 \quad \text{where} \quad k_\beta := (-\omega_{\beta, \vec{k}}, \vec{k}). \quad (3.86)$$

The representation (3.85) has the convenient feature that the wave function at time t is given by

$$\psi_\beta^u(t, \vec{x}) = \int \frac{d^4k}{(2\pi)^4} \hat{\psi}_\beta^u(k) e^{-ikx} = e^{i\omega_{\beta, \vec{k}}t} \int \frac{d^3k}{(2\pi)^3} \chi_\beta(\vec{k}) e^{i\vec{k}\vec{x}},$$

showing that $\chi_\beta(\vec{k})$ simply is the *spatial* Fourier transform of the Dirac wave function at time zero.

Lemma 3.4.7. *The integral (3.84), can be written as*

$$J = \sum_{\alpha, \beta=1}^3 J_{\alpha, \beta}, \quad (3.87)$$

where the $J_{\alpha, \beta}$ are given by

$$J_{\alpha, \beta} = \lim_{\eta \searrow 0} \operatorname{Im} \int \frac{d^4k}{(2\pi)^4} \prec \chi_\alpha(\vec{k}) \frac{i}{k^0 + \omega_{\alpha, \vec{k}} + i\eta} | \hat{Q}(k) \chi_\beta(\vec{k}) \frac{-i}{k^0 + \omega_{\beta, \vec{k}} - i\eta} \succ. \quad (3.88)$$

Proof. We first rewrite (3.84) as

$$J = \lim_{\eta \searrow 0} \operatorname{Im} \int d^4x \int d^4y \prec \Theta(x^0) e^{-\eta x^0} \psi^u(x) | Q(x, y) \Theta(-y^0) e^{\eta y^0} \psi^u(y) \succ.$$

Since Q depends only on the difference vector $y - x$, the y -integration can be regarded as a convolution in position space. We now rewrite this convolution as a multiplication

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in momentum space. Setting

$$\hat{\psi}_\eta^\pm(k) := \int \Theta_\eta(\pm y^0) \psi^u(y) e^{iky} d^4y ,$$

where we introduced the “regularized Heaviside function”

$$\Theta_\eta(x) = \Theta(x) e^{-\eta x} ,$$

we obtain

$$J = \lim_{\eta \searrow 0} \operatorname{Im} \int_M d^4x \prec \Theta_\eta(x^0) \psi^u(x) | \mathcal{F}^{-1}(\hat{Q} \hat{\psi}_\eta^-)(x) \succ ,$$

where \mathcal{F}^{-1} denotes the inverse Fourier transformation. Plancherel’s theorem yields

$$J = \lim_{\eta \searrow 0} \operatorname{Im} \int \frac{d^4k}{(2\pi)^4} \prec \hat{\psi}_\eta^+(k) | \hat{Q}(k) \hat{\psi}_\eta^-(k) \succ . \quad (3.89)$$

We next compute $\hat{\psi}_\eta^\pm(k)$. Since multiplication in position space corresponds to convolution in momentum space, we know that

$$\hat{\psi}_\eta^\pm(k) = \int \frac{d\omega}{2\pi} \hat{\Theta}_\eta(\pm\omega) \hat{\psi}^u(k - (\omega, \vec{0})) . \quad (3.90)$$

Here the Fourier transformation of the regularized Heaviside function is computed by

$$\hat{\Theta}_\eta(\omega) = \int_{-\infty}^{\infty} \Theta_\eta(t) e^{i\omega t} dt = \frac{i}{\omega + i\eta} . \quad (3.91)$$

Using (3.91) and (3.85) in (3.90), we obtain

$$\hat{\psi}_\eta^\pm(k) = \left(\chi_\beta(\vec{k}) \frac{i}{\pm(k^0 + \omega_{\beta,\vec{k}}) + i\eta} \right)_{\beta=1,2,3} .$$

Using these formulas in (3.89) gives the result. \square

The next lemma shows that the summands for $\alpha \neq \beta$ drop out of (3.87).

Lemma 3.4.8. *The currents (3.88) satisfy the relation*

$$\sum_{\alpha \neq \beta} J_{\alpha,\beta} = 0 .$$

Proof. In the case $\alpha \neq \beta$, we know that $\omega_{\alpha,\vec{k}} \neq \omega_{\beta,\vec{k}}$, so that in (3.88) there are two single poles at $k^0 = -\omega_{\alpha,\vec{k}} - i\eta$ and $k^0 = -\omega_{\beta,\vec{k}} + i\eta$. This makes it possible to take the limit $\eta \rightarrow 0$ using the formula

$$\lim_{\eta \searrow 0} \frac{1}{x \pm i\eta} = \frac{\operatorname{PP}}{x} \mp i\pi \delta(x)$$

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(where PP denotes the principal value). We thus obtain

$$\begin{aligned}
J_{\alpha,\beta} &= -\text{Im} \int_M \frac{d^4 k}{(2\pi)^4} \frac{\text{PP}}{k^0 + \omega_{\alpha,\vec{k}}} \frac{\text{PP}}{k^0 + \omega_{\beta,\vec{k}}} \prec \chi_\alpha(\vec{k}) | \hat{Q}(k) \chi_\beta(\vec{k}) \succ \\
&\quad - \text{Im} \int_M \frac{d^4 k}{(2\pi)^4} \prec \chi_\alpha(\vec{k}) (-i\pi \delta(k^0 + \omega_{\alpha,\vec{k}})) | \hat{Q}(k) \chi_\beta(\vec{k}) \frac{\text{PP}}{k^0 + \omega_{\beta,\vec{k}}} \succ \\
&\quad - \text{Im} \int_M \frac{d^4 k}{(2\pi)^4} \prec \chi_\alpha(\vec{k}) \frac{\text{PP}}{k^0 + \omega_{\alpha,\vec{k}}} | \hat{Q}(k) \chi_\beta(\vec{k}) (i\pi \delta(k^0 + \omega_{\beta,\vec{k}})) \succ .
\end{aligned}$$

Carrying out the k^0 -integration in the last two lines gives

$$\begin{aligned}
J_{\alpha,\beta} &= -\text{Im} \int_M \frac{d^4 k}{(2\pi)^4} \frac{\text{PP}}{k^0 + \omega_{\alpha,\vec{k}}} \frac{\text{PP}}{k^0 + \omega_{\beta,\vec{k}}} \prec \chi_\alpha(\vec{k}) | \hat{Q}(k) \chi_\beta(\vec{k}) \succ \\
&\quad + \pi \text{Re} \int_M \frac{d^3 k}{(2\pi)^4} \prec \chi_\alpha(\vec{k}) | \hat{Q}(-\omega_{\alpha,\vec{k}}, \vec{k}) \chi_\beta(\vec{k}) \frac{\text{PP}}{-\omega_{\alpha,\vec{k}} + \omega_{\beta,\vec{k}}} \succ \\
&\quad + \pi \text{Re} \int_M \frac{d^3 k}{(2\pi)^4} \prec \chi_\alpha(\vec{k}) \frac{\text{PP}}{-\omega_{\beta,\vec{k}} + \omega_{\alpha,\vec{k}}} | \hat{Q}(-\omega_{\beta,\vec{k}}, \vec{k}) \chi_\beta(\vec{k}) \succ \\
&= -\text{Im} \int_M \frac{d^4 k}{(2\pi)^4} \frac{\text{PP}}{k^0 + \omega_{\alpha,\vec{k}}} \frac{\text{PP}}{k^0 + \omega_{\beta,\vec{k}}} \prec \chi_\alpha(\vec{k}) | \hat{Q}(k) \chi_\beta(\vec{k}) \succ \tag{3.92}
\end{aligned}$$

$$\begin{aligned}
&\quad + \pi \text{Re} \int_M \frac{d^3 k}{(2\pi)^4} \frac{\text{PP}}{\omega_{\alpha,\vec{k}} - \omega_{\beta,\vec{k}}} \\
&\quad \times \prec \chi_\alpha(\vec{k}) | \left(\hat{Q}(-\omega_{\beta,\vec{k}}, \vec{k}) - \hat{Q}(-\omega_{\alpha,\vec{k}}, \vec{k}) \right) \chi_\beta(\vec{k}) \succ . \tag{3.93}
\end{aligned}$$

Obviously, the contribution (3.92) is anti-symmetric when exchanging α and β . In the contribution (3.93), on the other hand, we can use the Dirac equation (3.86) together with (3.79) to rewrite the spin scalar product as

$$\prec \chi_\alpha(\vec{k}) | ((a+b)(m_\beta^2) - (a+b)(m_\alpha^2)) \chi_\beta(\vec{k}) \succ ,$$

and this vanishes by (3.80). This gives the result. \square

Using this lemma, our conserved integral (3.87) simplifies to

$$J = \sum_{\beta=1}^3 J_{\beta,\beta} . \tag{3.94}$$

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We now compute $J_{\beta,\beta}$. First,

$$\begin{aligned}
J_{\beta,\beta} &= \lim_{\eta \searrow 0} \operatorname{Im} \int_M \frac{d^4 k}{(2\pi)^4} \prec \chi_\beta(\vec{k}) \frac{i}{k^0 + \omega_{\beta,\vec{k}} + i\eta} | \hat{Q}(k) \chi_\beta(\vec{k}) \frac{-i}{k^0 + \omega_{\beta,\vec{k}} - i\eta} \succ \\
&= \lim_{\eta \searrow 0} \int_M \frac{d^4 k}{(2\pi)^4} \prec \chi_\beta(\vec{k}) | \hat{Q}(k) \frac{1}{2i} \left(\frac{-1}{(k^0 + \omega_{\beta,\vec{k}} - i\eta)^2} - \frac{-1}{(k^0 + \omega_{\beta,\vec{k}} + i\eta)^2} \right) \chi_\beta(\vec{k}) \succ \\
&\stackrel{(\star)}{=} - \lim_{\eta \searrow 0} \int \frac{d^3 k}{(2\pi)^2} \int_{-\infty}^{\infty} \frac{dq}{2\pi} \prec \chi_\beta(\vec{k}) | \hat{Q}(q - \omega_{\beta,\vec{k}}, \vec{k}) \frac{1}{2i} \left(\frac{1}{(q + i\eta)^2} - \frac{1}{(q - i\eta)^2} \right) \chi_\beta(\vec{k}) \succ \\
&= -2 \lim_{\eta \searrow 0} \int \frac{d^3 k}{(2\pi)^3} \prec \chi_\beta(\vec{k}) | \int_{-\infty}^{\infty} \frac{dq}{2\pi} \left(\hat{Q}(q - \omega_{\beta,\vec{k}}, \vec{k}) \frac{q\eta}{(q^2 + \eta^2)^2} \right) \chi_\beta(\vec{k}) \succ, \quad (3.95)
\end{aligned}$$

where in (\star) we introduced the variable $q = k^0 + \omega_{\beta,\vec{k}}$. We now use (3.81) to expand \hat{Q} for small q according to

$$\begin{aligned}
&\hat{Q}(q - \omega_{\beta,\vec{k}}, \vec{k}) \\
&= \hat{Q}(-\omega_{\beta,\vec{k}}, \vec{k}) + q \Theta(q) \partial_\omega^+ \hat{Q}(-\omega_{\beta,\vec{k}}, \vec{k}) + q \Theta(-q) \partial_\omega^- \hat{Q}(-\omega_{\beta,\vec{k}}, \vec{k}) + o(q)
\end{aligned}$$

(where $o(q)$ is the usual remainder term). Substituting this Taylor expansion into (3.95), the constant term of the expansion drops out because the integrand is odd. For the left and right derivatives, the integral can be carried out explicitly using that

$$\int_0^\infty \frac{q^2 \eta}{(q^2 + \eta^2)^2} dq = \frac{\pi}{4} = \int_{-\infty}^0 \frac{q^2 \eta}{(q^2 + \eta^2)^2} dq. \quad (3.96)$$

Thus, disregarding the remainder term, we obtain

$$J_{\beta,\beta} = -\frac{1}{4} \int \frac{d^3 k}{(2\pi)^3} \prec \chi_\beta(\vec{k}) | \left((\partial_\omega^+ + \partial_\omega^-) \hat{Q}(-\omega_{\beta,\vec{k}}, \vec{k}) \right) \chi_\beta(\vec{k}) \succ. \quad (3.97)$$

This formula corresponds to the result of Lemma 3.4.5 in our setting where $\hat{Q}(k)$ is not differentiable on the mass shells.

It remains to analyze the remainder term. Naively, the integrated remainder term is of the order η and should thus vanish in the limit $\eta \searrow 0$. This could indeed be proved if we knew for example that the function $\hat{Q}(\cdot - \omega_{\beta,\vec{k}}, \vec{k})$ is integrable. However, since \hat{Q} is only defined on the lower mass cone (see Definition 3.4.6), such arguments cannot be applied. Our method for avoiding this technical problem is to work with a convergence-generating factor with compact support in momentum space. To this end, we choose a non-negative test function $\hat{g} \in C_0^\infty((-1, 1))$ with $\hat{g}(-\omega) = \hat{g}(\omega)$ for all $\omega \in \mathbb{R}$ and $\int_{\mathbb{R}} \hat{g}(\omega) d\omega = 2\pi$. For given $\sigma > 0$ we set

$$\hat{g}_\sigma(\omega) = \frac{1}{\sigma} \hat{g}\left(\frac{\omega}{\sigma}\right) \quad \text{and} \quad g_\sigma(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \hat{g}_\sigma(\omega) e^{-i\omega t}.$$

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In the limit $\sigma \searrow 0$, the functions $g_\sigma(t)$ go over to the constant function one.

Lemma 3.4.9. *Replacing (3.84) by*

$$J = \lim_{\sigma \searrow 0} \int_{t \geq 0} d^4 x \int_{t < 0} d^4 y g_\sigma(x^0) g_\sigma(y^0) \operatorname{Im} (\prec \psi^u(y) | Q(y, x) \psi^u(x) \succ), \quad (3.98)$$

the resulting function J is of the form (3.94) with $J_{\beta, \beta}$ as given by (3.97).

Proof. Again rewriting (3.98) in momentum space and using that \hat{g} has compact support, one sees that the resulting integrand of $J_{\alpha, \beta}$ is well-defined for any \vec{k} for sufficiently small σ . In order to relate the functions g_σ in (3.98) to the factor $e^{-\eta x^0 + \eta y^0}$ in (3.84), it is most convenient to work with the Laplace transform. Thus we represent the functions g_σ in (3.98) for $x^0 > 0$ and $y^0 < 0$ as

$$g_\sigma(x^0) = \frac{1}{\sigma} \int_0^\infty h\left(\frac{\eta}{\sigma}\right) e^{-\eta x^0} d\eta \quad \text{and} \quad g_\sigma(y^0) = \frac{1}{\sigma} \int_0^\infty h\left(\frac{\tilde{\eta}}{\sigma}\right) e^{\tilde{\eta} y^0} d\tilde{\eta},$$

where h is the inverse Laplace transform of g (for basics on the Laplace transform see for example [Dav02]). A straightforward computation shows that the result of Lemma 3.4.7 remains valid with the obvious replacements. The computation of $J_{\beta, \beta}$, on the other hand, needs to be modified as follows. Formula (3.95) remains valid after the replacement

$$\lim_{\eta \searrow 0} \cdots \frac{q \eta}{(q^2 + \eta^2)^2} \longrightarrow \lim_{\sigma \searrow 0} \frac{1}{\sigma^2} \int_0^\infty h\left(\frac{\eta}{\sigma}\right) d\eta \int_0^\infty h\left(\frac{\tilde{\eta}}{\sigma}\right) d\tilde{\eta} \cdots \frac{q(\eta + \tilde{\eta})}{2(q^2 + \eta^2)(q^2 + \tilde{\eta}^2)}.$$

Substituting the Taylor expansion of \hat{Q} , the first integral in (3.96) is to be replaced by the integral

$$\int_0^\infty \frac{q^2(\eta + \tilde{\eta})}{2(q^2 + \eta^2)(q^2 + \tilde{\eta}^2)} d\eta d\tilde{\eta} = \frac{\pi}{4}$$

(and similarly for the second integral in (3.96)). In this way, one again obtains (3.97), but now the remainder term vanishes in the limit $\sigma \searrow 0$. \square

We now compute $J_{\beta, \beta}$ more explicitly.

Lemma 3.4.10. *The currents $J_{\beta, \beta}$ given by (3.97) can be written as*

$$J_{\beta, \beta} = -\frac{m_\beta c_\beta}{2} \int_{\mathbb{R}^3} \prec \psi_\beta^u(x) | \gamma^0 \psi_\beta^u(x) \succ d^3 x \quad (3.99)$$

with the constants c_β as in (3.83).

Proof. Using (3.79) and applying the chain rule for semi-derivatives, we obtain

$$\partial_\omega^\pm \hat{Q}(-\omega_{\beta, \vec{k}}, \vec{k}) = -2\omega_{\beta, \vec{k}} \left(\partial_\omega^\pm a(k_-^2) \frac{k_-}{|k_-|} + \partial_\omega^\pm b(k_-^2) \right) + a(k_-^2) \frac{\partial}{\partial k^0} \left(\frac{k}{|k|} \right) \Big|_{k=k_-}, \quad (3.100)$$

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where we set $k_- = (-\omega_{\beta, \vec{k}}, \vec{k})$ and $|k_-| = \sqrt{k_-^2} = m_\beta$. This formula can be further simplified when taking the expectation value with the spinor $\chi_\beta(\vec{k})$: In the last summand in (3.100), we first compute the k -derivative,

$$\frac{\partial}{\partial k^0} \frac{k_-}{|k_-|} \Big|_{k=k_-} = \frac{\gamma^0}{m_\beta} - \frac{k_-}{|k_-|^3} k_-^0.$$

Taking the expectation value with the spinor $\chi_\beta(\vec{k})$ and using the Dirac equation

$$(k_- - m_\beta) \chi_\beta(\vec{k}) = 0,$$

we obtain the relations

$$\prec \chi_\beta(\vec{k}) | k_- \chi_\beta(\vec{k}) \succ = m_\beta \prec \chi_\beta(\vec{k}) | \chi_\beta(\vec{k}) \succ \quad (3.101)$$

$$2m_\beta \prec \chi_\beta(\vec{k}) | \gamma^0 \chi_\beta(\vec{k}) \succ = \prec \chi_\beta(\vec{k}) | \{k_-, \gamma^0\} \chi_\beta(\vec{k}) \succ = -2\omega_{\beta, \vec{k}} \prec \chi_\beta(\vec{k}) | \chi_\beta(\vec{k}) \succ. \quad (3.102)$$

In this way, the last summand in (3.100) gives zero. In the remaining first summand in (3.100), we again employ the Dirac equation $(k_- - m_\beta) \chi_\beta(\vec{k}) = 0$ to obtain

$$\left(\partial_\omega^\pm a(k_-^2) \frac{k_-}{|k_-|} + \partial_\omega^\pm b(k_-^2) \right) \chi_\beta(\vec{k}) = \partial_\omega^\pm \left(a(k_-^2) + b(k_-^2) \right) \chi_\beta(\vec{k}).$$

We conclude that

$$J_{\beta, \beta} = \frac{1}{2} c_\beta \int \frac{d^3 k}{(2\pi)^3} \omega_{\beta, \vec{k}} \prec \chi_\beta(\vec{k}) | \chi_\beta(\vec{k}) \succ \quad (3.103)$$

with c_β as in (3.83). We finally use (3.102) and apply Plancherel's theorem. \square

Combining Lemma 3.4.9 and Lemma 3.4.10 gives the conservation law (3.61). This concludes the proof of Theorem 3.4.3.

3.4.3 Clarifying Remarks

The following remarks explain and clarify various aspects of the above constructions and results.

Remark 3.4.11. (DIFFERENTIABILITY OF VARIATIONS) We now explain in which sense the differentiability assumption on the function $\ell \circ \Phi$ in Theorem 3.4.2 is satisfied. First, the above computations show that, working with the specific form of \hat{Q} in the continuum limit, the τ -derivative exists and is finite. However, this does not necessarily imply that for any UV regularization, the corresponding local minimizers $(\mathcal{H}, \mathcal{F}, \rho^\varepsilon)$ also satisfy the differentiability assumptions on the function $\ell \circ \Phi$ in Theorem 3.4.2. Indeed, thinking of a lattice regularization, we expect that the function $\ell \circ \Phi$ with Φ according to (3.58) and (3.57) will typically *not* be continuously differentiable in τ (because in this case, ℓ is a sum of terms involving the Lagrangian, which is only Lipschitz continuous). In order to bypass this technical problem, for a given local minimizer $(\mathcal{H}, \mathcal{F}, \rho^\varepsilon)$ one can modify Φ

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such as to obtain a variation Φ^ε for which the function $\ell \circ \Phi^\varepsilon$ is continuously differentiable in τ . For this modified variation, we have the conservation law of Theorem 3.4.2. The strategy is to choose the Φ^ε for every $\varepsilon > 0$ in such a way that in the limit $\varepsilon \searrow 0$, the variations converge in a suitable weak topology to the variation Φ_τ as given by (3.58) and (3.57). In non-technical terms, we modify Φ_τ by “microscopic fluctuations” in such a way that the functions $\ell \circ \Phi^\varepsilon$ become differentiable in τ for all $\varepsilon > 0$. In the limit $\varepsilon \searrow 0$, the microscopic fluctuations should drop out to give Theorem 3.4.3.

At present, this procedure cannot be carried out because, so far, no local minimizers $(\mathcal{H}, \mathcal{F}, \rho^\varepsilon)$ have been constructed which describe regularized Dirac sea configurations. The difficulty is to arrange the regularization in such a way that the EL equations are satisfied without error terms. A first step towards the construction of such “optimal regularizations” is given in [Fin08]. \diamond

Remark 3.4.12. (WEIGHT FACTORS) As explained in [Fin08, Section 2 and Appendix A], one may introduce positive weight factors ρ_β into the ansatz (3.60),

$$P(x, y) = \sum_{\beta=1}^3 \rho_\beta \int \frac{d^4 k}{(2\pi)^4} (\not{k} + m_\beta) \delta(k^2 - m_\beta^2) e^{-ik(x-y)}.$$

The above analysis immediately extends to this situation simply by inserting suitable factors of ρ_β into all equations. In particular, the resulting conserved quantity (3.99) becomes

$$J_{\beta,\beta} = -\frac{\rho_\beta m_\beta c_\beta}{2} \int_{\mathbb{R}^3} \prec \psi_\beta^u(x) | \gamma^0 \psi_\beta^u(x) \succ d^3 x.$$

Consequently, the conserved current in (3.61) is to be modified to

$$\sum_{\beta=1}^3 \rho_\beta m_\beta c_\beta \int_{t=\text{const}} \prec \psi_\beta^u(x) | \gamma^0 \psi_\beta^u(x) \succ d^3 x.$$

The role of the weight factors in the interacting case will be explained in the next remark. \diamond

Remark 3.4.13. (INTERACTING SYSTEMS) We point out that for the derivation of Theorem 3.4.3, we worked with the vacuum Dirac equations (3.59), so that no interaction is present. In particular, the generations have an independent dynamics, implying that current conservation holds separately for each generation, i.e.

$$\int_{t=t_0} \prec \psi_\beta^u(x) | \gamma^0 \psi_\beta^u(x) \succ d^3 x = \int_{t=t_1} \prec \psi_\beta^u(x) | \gamma^0 \psi_\beta^u(x) \succ d^3 x \quad \text{for all } \beta = 1, 2, 3. \quad (3.104)$$

Let us now discuss the typical situation of a scattering process in which the Dirac equations (3.59) only hold asymptotically as $t \rightarrow \pm\infty$. In this case, choosing Ω so large that it contains the interaction region, one can compute the surface layer integrals again for the free Dirac equation to obtain the conservation law (3.61), where t_0 lies in the

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past and t_1 in the future of the interaction region. In this way, the conservation law of Theorem 3.4.3 immediately extends to interacting systems.

In this interacting situation, current conservation no longer holds for each generation separately (thus (3.104) is violated). Instead, as a consequence of the Dirac dynamics, only the total charge

$$\sum_{\beta=1}^3 \int_{t=\text{const}} \prec \psi_{\beta}^u(x) | \gamma^0 \psi_{\beta}^u(x) \succ d^3x \quad (3.105)$$

is conserved. In order for this conservation law to be compatible with (3.61), we need to impose that

$$m_{\alpha} c_{\alpha} = m_{\beta} c_{\beta} \quad \text{for all } \alpha, \beta = 1, 2, 3. \quad (3.106)$$

This is a mathematical consistency condition which gives information on the possible form of the distribution $\hat{Q}(k)$ in the continuum limit (as specified in Definition 3.4.6 above). If weight factors are present (see Remark 3.4.12 above), this consistency condition must be modified to

$$\rho_{\alpha} m_{\alpha} c_{\alpha} = \rho_{\beta} m_{\beta} c_{\beta} \quad \text{for all } \alpha, \beta = 1, 2, 3. \quad (3.107)$$

The conditions (3.106) and (3.107) are crucial for the future project of extending the state stability analysis in [FH09] to systems involving neutrinos. \diamond

Remark 3.4.14. (NORMALIZATION OF THE FERMIONIC PROJECTOR) The conservation law of Theorem 3.4.3 has an important implication for the normalization of the fermionic projector, as we now explain. As worked out in detail in [FT14], there are two alternative normalization methods for the fermionic projector: the spatial normalization and the mass normalization. In [FT14, Section 2.2] the advantages of the spatial normalization are discussed, but no decisive argument in favor of one of the normalization methods is given. Theorem 3.4.3 decides the normalization problem in favor of the spatial normalization. Namely, this theorem shows that the dynamics as described by the causal action principle gives rise to a conservation law which in the continuum limit reduces to the spatial integrals (3.61). As explained in Remark 3.4.12 above, the mathematical consistency to the Dirac dynamics implies that (3.61) coincides with the conserved total charge (3.105). The resulting conservation law is compatible with the spatial normalization, but contradicts the mass normalization. We conclude that the spatial normalization of the fermionic projector is indeed the correct normalization method which reflects the intrinsic conservation laws of the causal fermion system. \diamond

3.5 Example: Conservation of Energy-Momentum

The conservation laws in Theorem 3.3.7 also give rise to the *conservation of energy and momentum*, as will be worked out in this section.

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3.5.1 Generalized Killing Symmetries and Conservation Laws

In the classical Noether theorem, the conservation laws of energy and momentum are a consequence of space-time symmetries described most conveniently with the notion of Killing fields. Therefore, one of our tasks is to extend this notion to the setting of causal fermion systems. In preparation, we recall the procedure in the classical Noether theorem from a specific point of view: In the notion of a Killing field, one distinguishes the background geometry from the additional particles and fields. The background geometry must have a symmetry as described by the Killing equation. The additional particles and fields, however, do not need to have any symmetries. Nevertheless, one can construct a symmetry of the whole system by actively transporting the particles and fields along the flow lines of the Killing field. The conservation law corresponding to this symmetry transformation gives rise to the conservation of energy and momentum.

In a causal fermion system, there is no clear-cut distinction between the background geometry and the particles and fields of the system, because all of these structures are inherent in the underlying causal fermion system and mutually influence each other via the causal action principle. Therefore, instead of working with a symmetry of the background geometry, we shall work with the notion of an approximate symmetry. By actively transforming those physical wave functions which do not respect the symmetry, such an approximate symmetry again gives rise to an exact symmetry transformation, to which our Noether-like theorems apply.

More precisely, one begins with a C^1 -family of transformations $(f_\tau)_{\tau \in (-\tau_{\max}, \tau_{\max})}$ of space-time,

$$f_\tau : M \rightarrow M \quad \text{with} \quad f_0 = \mathbb{1} ,$$

which preserve the universal measure in the sense that $(f_\tau)_* \rho = \rho$. This family can be regarded as the analog of the flow in space-time along a classical Killing field. Moreover, one considers a family of unitary transformations $(\mathcal{U}_\tau)_{\tau \in (-\tau_{\max}, \tau_{\max})}$ on \mathcal{H} with the property that

$$\mathcal{U}_{-\tau} \mathcal{U}_\tau = \mathbb{1} \quad \text{for all } \tau \in (-\tau_{\max}, \tau_{\max}) , \quad (3.108)$$

and defines the variation

$$\Phi : (-\tau_{\max}, \tau_{\max}) \times M \rightarrow \mathcal{F} , \quad \Phi(\tau, x) := \mathcal{U}_\tau x \mathcal{U}_\tau^{-1} . \quad (3.109)$$

Combining these transformations should give rise to an *approximate symmetry* of the wave evaluation operator (3.65) in the sense that if we compare the transformation of the space-time point with the unitary transformation by setting

$$E_\tau(u, x) := (\Psi u)(f_\tau(x)) - (\Psi \mathcal{U}_\tau^{-1} u)(x) \quad (x \in M, u \in \mathcal{H}) , \quad (3.110)$$

then the operator $E_\tau : \mathcal{H} \rightarrow C^0(M, SM)$ should be so small that the first variation is well-defined in the continuum limit (for details see Section 3.5.2 below). There are various ways in which this smallness condition could be formulated. We choose a simple method which is most convenient for our purposes.

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Definition 3.5.1. The transformation $(f_\tau)_{\tau \in (-\tau_{\max}, \tau_{\max})}$ is called a **Killing symmetry with finite-dimensional support** of the causal fermion system if it is a symmetry of the universal measure that preserves the trace (see Definition 3.3.8) and if there exists a finite-dimensional subspace $K \subset \mathcal{H}$ and a family of unitary operators $(\mathcal{U}_\tau)_{\tau \in (-\tau_{\max}, \tau_{\max})}$ with the property (3.108) such that

$$E_\tau(u, x) = 0 \quad \text{for all } u \in K^\perp \text{ and } x \in M. \quad (3.111)$$

We now formulate a general conservation law.

Theorem 3.5.2. Let ρ be a local minimizer (see Definition 3.3.3) and $(f_\tau)_{\tau \in (-\tau_{\max}, \tau_{\max})}$ be a Killing symmetry of the causal fermion system. Then the following conservation law holds:

$$\begin{aligned} \frac{d}{d\tau} \int_{\Omega} d\rho(x) \int_{M \setminus \Omega} d\rho(y) \left(\mathcal{L}_\kappa(f_\tau(x), y) - \mathcal{L}_\kappa(x, f_\tau(y)) \right. \\ \left. - \mathcal{L}_\kappa(\Phi_\tau(x), y) + \mathcal{L}_\kappa(x, \Phi_\tau(y)) \right) \Big|_{\tau=0} = 0. \end{aligned} \quad (3.112)$$

Proof. Again using Lemma 3.4.1, we know that the variation (3.109) is a symmetry of the Lagrangian. Hence

$$\begin{aligned} \int_M d\rho(x) \int_{\Omega} d\rho(y) \mathcal{L}_\kappa(\Phi_\tau(x), y) &= \int_M d\rho(x) \int_{\Omega} d\rho(y) \mathcal{L}_\kappa(x, \Phi_{-\tau}(y)) \\ &= \int_{\Omega} d\rho(y) \ell(\Phi_\tau(y)). \end{aligned}$$

Using this equation in Proposition 3.3.5, we obtain

$$0 = \frac{d}{d\tau} \int_{\Omega} d\rho(x) \int_{M \setminus \Omega} d\rho(y) \left(\mathcal{L}_\kappa(\Phi_\tau(x), y) - \mathcal{L}_\kappa(x, \Phi_\tau(y)) \right) \Big|_{\tau=0}. \quad (3.113)$$

For the transformations f_τ , on the other hand, we have the relations

$$\int_M \mathcal{L}_\kappa(f_\tau(x), y) d\rho(x) = \int_{\mathcal{F}} \mathcal{L}_\kappa(z, y) d((f_\tau)_*\rho)(z) = \int_M \mathcal{L}_\kappa(x, y) d\rho(x),$$

where in the last step we used that f_τ is a symmetry of the universal measure. Since f_τ also preserves the trace, it is a generalized integrated symmetry (see Definition 3.3.6). Applying Theorem 3.3.7, we obtain

$$\frac{d}{d\tau} \int_{\Omega} d\rho(x) \int_{M \setminus \Omega} d\rho(y) \left(\mathcal{L}_\kappa(f_\tau(x), y) - \mathcal{L}_\kappa(x, f_\tau(y)) \right) \Big|_{\tau=0} = 0. \quad (3.114)$$

Subtracting (3.114) from (3.113) gives the result. \square

We remark that the vector field $w := \delta f$ is tangential to M and describes a transformation of the space-time points. The variation $\delta\Phi$, on the other hand, is a vector

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field in \mathcal{F} along M . It will in general not be tangential to M . The difference vector field $v := w - \delta\Phi$ can be understood as an active transformation of all the objects in space-time which do not have the space-time symmetry (similar to the parallel transport of the particles and fields along the flow lines of the Killing field in the classical Noether theorem as described above). The variation of the integrand in (3.112) can be rewritten as a variation in the direction v ; for example,

$$\left. \frac{d}{d\tau} \left(\mathcal{L}_\kappa(f_\tau(x), y) - \mathcal{L}_\kappa(\Phi_\tau(x), y) \right) \right|_{\tau=0} = \delta_{v(x)} \mathcal{L}_\kappa(x, y) .$$

Expressing v in terms of the operator E in (3.110) and using (3.111) will show that v is indeed so small (in a suitable sense) that the corresponding variation of the Lagrangian will be well-defined and finite.

3.5.2 Correspondence to the Dirac Energy-Momentum Tensor

In order to get the connection to the conservation of energy and momentum, as in Section 3.4.2 we consider the vacuum Dirac equation and the limiting case that Ω exhausts the region between two Cauchy surfaces $t = t_0$ and $t = t_1$ (see Figure 3.2). Recall that the energy-momentum tensor of a Dirac wave function ψ is given by

$$T_{jk} = \frac{1}{2} \operatorname{Re} (\langle \psi | \gamma_j i \partial_k \psi \rangle + \langle \psi | \gamma_k i \partial_j \psi \rangle) = - \operatorname{Im} \langle \psi | \gamma_{(j} \partial_{k)} \psi \rangle .$$

We consider the situation of the vacuum Dirac sea with a finite number of holes describing the anti-particle states $\phi_1, \dots, \phi_{n_a}$ (for the description of particle states see again Section 3.7). The effective energy-momentum tensor is minus the sum of the energy-momentum tensors of all the anti-particle states. Thus for a fixed value of the generation index β , we set

$$(T_\beta)_{jk} = \sum_{i=1}^{n_a} \operatorname{Im} \langle \phi_{i,\beta} | \gamma_{(j} \partial_{k)} \phi_{i,\beta} \rangle .$$

In order to treat the generations, as in Theorem 3.4.3 we take a linear combination involving the non-negative constants c_β introduced in (3.83).

Theorem 3.5.3. (Energy conservation) *Let $(\mathcal{H}, \mathcal{F}, \rho^\varepsilon)$ be local minimizers of the causal action describing the Minkowski vacuum (3.60) together with particles and anti-particles. Considering the limiting procedure explained in Figure 3.2 and taking the continuum limit, the conservation law of Theorem 3.5.2 goes over to*

$$\sum_{\beta=1}^3 m_\beta c_\beta \int_{t=t_0} d^3x (T_\beta)_0^0 d^3x = \sum_{\beta=1}^3 m_\beta c_\beta \int_{t=t_1} d^3x (T_\beta)_0^0 d^3x .$$

This theorem can be extended immediately to energy-momentum conservation on general Cauchy surfaces:

Corollary 3.5.4. (Energy-momentum conservation on Cauchy surfaces)

Let $\mathcal{N}_0, \mathcal{N}_1$ be two Cauchy surfaces in Minkowski space, where \mathcal{N}_1 lies to the future of \mathcal{N}_0 . Then, under the assumptions of Theorem 3.5.3, the conservation law of Theorem 3.5.2 goes over to the conservation law for the energy and momentum integrals

$$\sum_{\beta=1}^3 m_{\beta} c_{\beta} \int_{\mathcal{N}_0} (T_{\beta})_k^j \nu_j d\mu_{\mathcal{N}_0} = \sum_{\beta=1}^3 m_{\beta} c_{\beta} \int_{\mathcal{N}_1} (T_{\beta})_k^j \nu_j d\mu_{\mathcal{N}_1}, \quad (3.115)$$

where ν again denotes the future-directed normal and $k \in \{0, \dots, 3\}$.

Proof. We use similar arguments as in the proof of Corollary 3.4.4. More precisely, the conservation of classical energy implies that

$$\int_{t=t_0} d^3x (T_{\beta})_0^0 d^3x = \int_{\mathcal{N}} (T_{\beta})_0^j \nu_j d\mu_{\mathcal{N}}.$$

This gives (3.115) in the case $k = 0$. Applying a Lorentz boost, we obtain

$$\sum_{\beta=1}^3 m_{\beta} c_{\beta} \int_{\mathcal{N}_0} (T_{\beta})_k^j \nu_j K^k d\mu_{\mathcal{N}_0} = \sum_{\beta=1}^3 m_{\beta} c_{\beta} \int_{\mathcal{N}_1} (T_{\beta})_k^j \nu_j K^k d\mu_{\mathcal{N}_1},$$

where K is the Killing field obtained by applying the Lorentz boost to the vector field ∂_t . This gives the result. \square

These results shows that the conservation laws of energy and momentum correspond to more general conservation laws in the setting of causal fermion systems.

The remainder of this section is devoted to the proof of Theorem 3.5.3. Let $(\mathcal{H}, \mathcal{F}, \rho^{\varepsilon})$ be a regularized vacuum Dirac sea configuration together with anti-particles (for details see [Fin16b, Sections 1.2 and 3.4]). Then, possibly after extending the particle space (see [Fin16b, Remark 1.2.2]), we can decompose the wave evaluation operator Ψ as

$$\Psi = \Psi^{\text{vac}} + \Delta\Psi, \quad (3.116)$$

where Ψ^{vac} is the wave evaluation operator of the completely filled Dirac sea (see [Fin16b, §1.1.4] and the operator $\Psi(x) = e_x^{\varepsilon}$ in [Fin16b, §1.2.4]), and $\Delta\Psi$ describes the holes. The fact that the number of anti-particles is finite implies that the operator $\Delta\Psi$ is trivial on the orthogonal complement of a finite-dimensional subspace of \mathcal{H} , which we denote by K ,

$$\Delta\Psi u = 0 \quad \text{for all } u \in K^{\perp} \subset \mathcal{H}. \quad (3.117)$$

We now choose $(f_{\tau})_{\tau \in \mathbb{R}}$ as the time translations, i.e.

$$f_{\tau} : \mathcal{M} \rightarrow \mathcal{M}, \quad f_{\tau}(t, x_1, x_2, x_3) = (t + \tau, x_1, x_2, x_3)$$

(for the identification of \mathcal{M} with $M := \text{supp } \rho$ see [Fin16b, Section 1.2]). Since the Lebesgue measure d^4x is translation invariant, it clearly is invariant under the action

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of f_τ . Constructing the universal measure as the push-forward (see [Fin16b, §1.2.1]), it follows immediately that f_τ is a symmetry of the universal measure.

Since Ψ^{vac} is composed of plane-wave solutions of the Dirac equation, on which the time translation operator acts by multiplication with a phase, the operator f_τ can be represented by a unitary transformation in \mathcal{H} . More precisely, choosing the operator \mathcal{U}_τ as the multiplication operator in momentum space $\hat{\mathcal{U}}(k) = e^{ik^0\tau}$, we have the relation

$$(\Psi^{\text{vac}} u)(f_\tau(x)) = (\Psi^{\text{vac}} \mathcal{U}_\tau^{-1} u)(x) \quad \text{for all } x \in M, u \in \mathcal{H}. \quad (3.118)$$

Using (3.116) and (3.118) in (3.110), we conclude that

$$E_\tau(u, x) := (\Delta \Psi u)(f_\tau(x)) - (\Delta \Psi \mathcal{U}_\tau^{-1} u)(x) \quad \text{for all } x \in M, u \in \mathcal{H}.$$

The assumption (3.117) implies that $(f_\tau)_{\tau \in \mathbb{R}}$ is indeed a Killing symmetry with finite-dimensional support (see Definition 3.5.1).

In order to simplify the setting, we note that a unitary transformation \mathcal{U}_τ was already used in Section 3.4 to obtain corresponding conserved currents (see (3.57) and Theorem 3.4.2). This means that the first variations of \mathcal{U}_τ on the finite-dimensional subspace K give rise to a linear combination of the corresponding conserved currents. With this in mind, we may in what follows assume that \mathcal{U}_τ is trivial on K ,

$$\mathcal{U}_\tau|_K = \mathbb{1}_K. \quad (3.119)$$

Modifying \mathcal{U}_τ in this way corresponds to going over to a new conservation law, which is obtained from the original conservation law by subtracting a linear combination of electromagnetic currents.

For the computations, it is most convenient to work again with the kernel of the fermionic projector. Using (3.116), we decompose it as

$$P(x, y) = P^{\text{vac}}(x, y) + \Delta P(x, y),$$

where

$$\begin{aligned} P^{\text{vac}}(x, y) &= -\Psi^{\text{vac}}(x) \Psi^{\text{vac}}(y)^* \\ \Delta P(x, y) &= -\Psi^{\text{vac}}(x) (\Delta \Psi)(y)^* - (\Delta \Psi)(x) \Psi^{\text{vac}}(y)^* - (\Delta \Psi)(x) (\Delta \Psi)(y)^*. \end{aligned}$$

Since $\Delta \Psi$ vanishes on the complement of the finite-dimensional subspace K , the kernel $\Delta \Psi$ is composed of a finite number of Dirac wave functions, i.e.

$$\Delta P(x, y) = \sum_{i,j=1}^{n_a} c_{ij} \phi_i(x) \overline{\phi_j(y)}$$

with $n_a \in \mathbb{N}$ and $\overline{c_{ij}} = c_{ji}$. Diagonalizing the Hermitian matrix (c_{ij}) by a basis transfor-

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mation, we can write $\Delta P(x, y)$ as

$$\Delta P(x, y) = \sum_{i=1}^{n_a} c_i \phi_i(x) \overline{\phi_i(y)}$$

with real-valued coefficients c_i . Since we only consider first order variations of the Lagrangian, by linearity we may restrict attention to one of the summands. Thus it suffices to consider the case

$$\Delta P(x, y) = \psi(x) \overline{\psi(y)},$$

where ψ is a negative-frequency solution of the Dirac equation.

Now the first variation of the Lagrangian can be computed similar as in (3.72) to obtain

$$\begin{aligned} & \frac{d}{d\tau} \int_{\Omega} d\rho(x) \int_{M \setminus \Omega} d\rho(y) \left(\mathcal{L}_{\kappa}(f_{\tau}(x), y) - \mathcal{L}_{\kappa}(\Phi_{\tau}(x), y) \right) \Big|_{\tau=0} \\ &= \int_{\Omega} d\rho(x) \int_{M \setminus \Omega} d\rho(y) \left(\text{Tr}_{S_y} (Q(y, x) \delta_{v(x)} P(x, y)) + \text{Tr}_{S_x} (Q(x, y) \delta_{v(x)} P(y, x)) \right), \end{aligned}$$

where

$$\delta_{v(x)} P(x, y) := \frac{d}{d\tau} \left(P(f_{\tau}(x), y) - P(\Phi_{\tau}(x), y) \right) \Big|_{\tau=0}.$$

Since $P^{\text{vac}}(x, y)$ has the Killing symmetry (3.118), the variation of $P(x, y)$ simplifies to

$$\begin{aligned} \delta_{v(x)} P(x, y) &= \frac{d}{d\tau} \left(\Delta P(f_{\tau}(x), y) - \Delta P(\Phi_{\tau}(x), y) \right) \Big|_{\tau=0} \\ &= \frac{d}{d\tau} \Delta P(f_{\tau}(x), y) \Big|_{\tau=0} = \frac{d}{d\tau} \left(\psi(f_{\tau}(x)) \overline{\psi(y)} \right) \Big|_{\tau=0} =: (\partial_t \psi)(x) \overline{\psi(y)}, \end{aligned}$$

where in the last line we used (3.119). Using these relations in (3.112), we obtain the conservation law

$$\begin{aligned} 0 &= \int_{\Omega} d\rho(x) \int_{M \setminus \Omega} d\rho(y) \left(\text{Tr}_{S_y} (Q(y, x) (\partial_t \psi)(x) \overline{\psi(y)}) + \text{Tr}_{S_x} (Q(x, y) \psi(y) \overline{(\partial_t \psi)(x)}) \right. \\ &\quad \left. - \text{Tr}_{S_y} (Q(y, x) \psi(x) \overline{(\partial_t \psi)(y)}) - \text{Tr}_{S_x} (Q(x, y) (\partial_t \psi)(y) \overline{\psi(x)}) \right) \\ &= 2 \text{Re} \int_{\Omega} d\rho(x) \int_{M \setminus \Omega} d\rho(y) \left(\prec \psi(y) | Q(y, x) (\partial_t \psi)(x) \succ - \prec \psi(x) | Q(x, y) (\partial_t \psi)(y) \succ \right). \end{aligned}$$

Next, we consider the limiting case where Ω exhausts the region between two Cauchy surfaces $t = t_0$ and $t = t_1$ (see Figure 3.2). We thus obtain a conserved current J which for example at time $t = 0$ is given by

$$J = \frac{1}{2} \text{Re} \int_{t \leq 0} d^4 x \int_{t > 0} d^4 y \left(\prec \psi(y) | Q(y, x) (\partial_t \psi)(x) \succ - \prec \psi(x) | Q(x, y) (\partial_t \psi)(y) \succ \right).$$

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This equation is similar to (3.73) and can be analyzed in exactly the same manner. Indeed, regularizing the Heaviside functions and applying Plancherel, we again obtain (3.92) and (3.93), with the only difference that an additional factor $\omega_{\beta, \vec{k}}$ appears. Thus, in analogy to (3.94) and (3.103) we obtain

$$J = - \sum_{\beta=1}^3 \frac{1}{2} c_{\beta} \int \frac{d^3 k}{(2\pi)^3} \omega_{\beta, \vec{k}}^2 \prec \chi_{\beta}(\vec{k}) | \chi_{\beta}(\vec{k}) \succ .$$

Applying (3.102) and using again Plancherel gives the result. This concludes the proof of Theorem 3.5.3.

3.6 Example: Symmetries of the Universal Measure

In this section we consider the conserved surface layer integrals corresponding to symmetries of the universal measure (see Theorem 3.2.5 and Corollary 3.3.10). We now explain why, under the assumption that Φ_{τ} is a bijection, these conserved surface layer integrals can be expressed merely in terms of the volumes of the sets $\Omega \setminus \Phi_{\tau}(\Omega)$ and $\Phi_{\tau}(\Omega) \setminus \Omega$. Our argument shows in particular that in the limiting case of Figure 3.2 when the boundary of Ω consists of two hypersurfaces, the conserved surface layer integrals do not give rise to any interesting conservation laws. Therefore, although the conservation laws of Theorem 3.2.5 and Corollary 3.3.10 give non-trivial information on the structure of a minimizing universal measure of a causal fermion system, they do not correspond to any conservation laws in Minkowski space.

The following argument applies for example to the situation considered in Section 3.4.2 that M can be identified with Minkowski space, and Ω is the past of a Cauchy surface. But the argument applies in a much more general setting. In particular, we do not need to assume that Ω is compact. We first rewrite the surface layer integral in (3.55) as

$$\begin{aligned} & \int_{\Omega} d\rho(x) \int_{M \setminus \Omega} d\rho(y) \left(\mathcal{L}_{\kappa}(\Phi_{\tau}(x), y) - \mathcal{L}_{\kappa}(x, \Phi_{\tau}(y)) \right) \\ &= \int_{\Omega} d\rho(x) \int_{M \setminus \Omega} d\rho(y) \left(\mathcal{L}_{\kappa}(\Phi_{\tau}(x), y) - \mathcal{L}_{\kappa}(x, y) \right) \\ & \quad + \int_{\Omega} d\rho(x) \int_{M \setminus \Omega} d\rho(y) \left(\mathcal{L}_{\kappa}(x, y) - \mathcal{L}_{\kappa}(x, \Phi_{\tau}(y)) \right) \\ &= \left(\int_{\Phi_{\tau}(\Omega)} - \int_{\Omega} \right) d\rho(x) \int_{M \setminus \Omega} d\rho(y) \mathcal{L}_{\kappa}(x, y) \\ & \quad + \int_{\Omega} d\rho(x) \left(\int_{M \setminus \Omega} - \int_{\Phi_{\tau}(M \setminus \Omega)} \right) d\rho(y) \mathcal{L}_{\kappa}(x, y) . \end{aligned}$$

Assuming that Φ_{τ} is a bijection, we can write the obtained differences of integrals as

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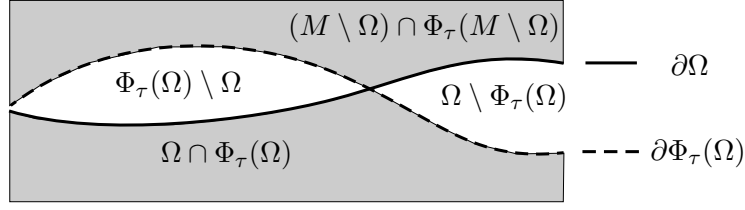


Figure 3.3: The surface layer integral corresponding to a symmetry of the universal measure.

illustrated in Figure 3.3,

$$\begin{aligned} \left(\int_{\Phi_\tau(\Omega)} - \int_{\Omega} \right) \cdots &= \left(\int_{\Phi_\tau(\Omega) \setminus \Omega} - \int_{\Omega \setminus \Phi_\tau(\Omega)} \right) \cdots \\ \left(\int_{M \setminus \Omega} - \int_{\Phi_\tau(M \setminus \Omega)} \right) \cdots &= \left(\int_{\Omega \setminus \Phi_\tau(\Omega)} - \int_{\Phi_\tau(\Omega) \setminus \Omega} \right) \cdots \end{aligned}$$

Using that (3.41) is symmetric with respect to exchange of its arguments (cf. (3.5)), we thus obtain

$$\begin{aligned} &\int_{\Omega} d\rho(x) \int_{M \setminus \Omega} d\rho(y) \left(\mathcal{L}_{\kappa}(\Phi_{\tau}(x), y) - \mathcal{L}_{\kappa}(x, \Phi_{\tau}(y)) \right) \\ &= \left(\int_{\Phi_{\tau}(\Omega) \setminus \Omega} - \int_{\Omega \setminus \Phi_{\tau}(\Omega)} \right) d\rho(x) \int_M d\rho(y) \mathcal{L}_{\kappa}(x, y) \\ &= \left(\int_{\Phi_{\tau}(\Omega) \setminus \Omega} - \int_{\Omega \setminus \Phi_{\tau}(\Omega)} \right) \ell(x) d\rho(x), \end{aligned}$$

where in the last step we used (3.45). In view of (3.47), the obtained integrand is constant. Therefore, the surface layer integral can indeed be expressed in terms of the volume of the sets $\Phi_{\tau}(\Omega) \setminus \Omega$ and $\Omega \setminus \Phi_{\tau}(\Omega)$. In particular, the surface layer integral does not capture any interesting dynamical information of the causal fermion system.

3.7 Conservation Laws and Microscopic Mixing

We conclude this chapter by again pointing out that the conservation laws of Theorem 3.4.2 and Theorem 3.5.2 hold for causal fermion systems without taking the continuum limit. In particular, these conservation laws also hold for regularized Dirac sea configurations if one analyzes the EL equations corresponding to the causal action principle without taking the limit $\varepsilon \searrow 0$ or in “quantum space-times” as outlined in Section 2.5.

In Theorem 3.4.3 and Theorem 3.5.3 we restricted attention to negative-frequency solutions of the Dirac equation. On a technical level, this was necessary because the operator \hat{Q} is only well-defined inside the *lower* mass cone, whereas it diverges outside the lower mass cone (see Definition 3.4.6 and [Fin07a, Section 5.6] or [Fin08]). In non-

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technical terms, this means that introducing Dirac particles into the causal fermion system makes the causal action infinitely large. But, as explained in detail in [Fin14, Section 3] and briefly in Section 2.7, the action becomes again finite if one introduces a so-called *microscopic mixing of the wave functions*. In other words, minimizing the causal action gives rise to the mechanism of microscopic mixing (for more details see [Fin16b, §1.5.3]). Microscopic mixing is also important for getting the connection to entanglement and second-quantized bosonic fields (see [Fin10b, Fin14] and Section 2.7).

If microscopic mixing is present, the conservation laws of Theorem 3.4.2 and Theorem 3.5.2 again give rise to conserved surface layer integrals. However, evaluating these surface layer integrals in Minkowski space is more involved because a homogenization procedure over the microstructure must be performed (in the spirit of [Fin14, Section 5.1]). Since these constructions are rather involved, we cannot give them here. However, even without entering the detailed constructions, the following argument shows that the conservation laws should apply to the particle states as well:

In an interacting system, a solution of the Dirac equation which at some initial time has negative frequency may at a later time have positive frequency (as in the usual pair production process). The conservation law of Theorem 3.4.2 implies that the surface layer integral at the later time coincides with that at the initial time. Using current conservation of the Dirac dynamics, we conclude that the surface layer at the later time again coincides with the surface integral of the Dirac current. Using arguments of this type, one sees that, no matter what the microscopic structure of space-time is, the conservation laws of Theorem 3.4.2 and Theorem 3.5.2 should apply similarly to positive-frequency solutions of the Dirac equation.

An explanation of the implications of the conservation laws constructed in this chapter on the foundations of quantum theory (in particular concerning the collapse of the wave function in the quantum mechanical measurement process) is given in Section 1.4.

Chapter 4

Hamiltonian Formulation and Linearized Field Equations

In this chapter, we give a formulation of the dynamics of causal fermion systems in terms of physical fields on space-time.

After generalizing causal variational principles to a class of lower semi-continuous Lagrangians on a smooth, possibly non-compact manifold, the corresponding Euler-Lagrange equations are derived (Section 4.1). In Section 4.2, we show under additional smoothness assumptions that the space of solutions of the Euler-Lagrange equations has the structure of a symplectic Fréchet manifold. The symplectic form is constructed as a surface layer integral which is shown to be invariant under the time evolution. In Section 4.3, the results and methods are extended to the lower semi-continuous setting. Evaluating the Euler-Lagrange equations weakly, we derive linearized field equations and the Hamiltonian time evolution. Finally, in Section 4.4, our constructions and results are illustrated in a detailed example on $\mathbb{R}^{1,1} \times S^1$, where a local minimizer is given by a measure supported on a two-dimensional lattice.

4.1 Causal Variational Principles and Causal Fermion Systems

This section provides the preliminaries needed for the construction of the Hamiltonian time evolution. After introducing causal variational principles in the non-compact setting (Section 4.1.1), we derive the corresponding Euler-Lagrange equations (Section 4.1.2) and introduce the concept of local minimizers (Section 4.1.3). The connection to the theory of causal fermion systems is established in Section 4.1.4.

4.1.1 Causal Variational Principles in the Non-Compact Setting

We now introduce causal variational principles in the non-compact setting (for the simpler compact setting see Section 3.1.2). Let \mathcal{F} be a (possibly non-compact) smooth manifold of dimension $m \geq 1$ and ρ a (positive) Borel measure on \mathcal{F} (the *universal measure*). Moreover, we are given a non-negative function $\mathcal{L} : \mathcal{F} \times \mathcal{F} \rightarrow \mathbb{R}_0^+$ (the *Lagrangian*) with the following properties:

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- (i) \mathcal{L} is symmetric: $\mathcal{L}(x, y) = \mathcal{L}(y, x)$ for all $x, y \in \mathcal{F}$.
- (ii) \mathcal{L} is lower semi-continuous, i.e. for all sequences $x_n \rightarrow x$ and $y_{n'} \rightarrow y$,

$$\mathcal{L}(x, y) \leq \liminf_{n, n' \rightarrow \infty} \mathcal{L}(x_n, y_{n'}) .$$

If the total volume $\rho(\mathcal{F})$ is finite, the *causal variational principle* is to minimize the action

$$\mathcal{S}(\rho) = \int_{\mathcal{F}} d\rho(x) \int_{\mathcal{F}} d\rho(y) \mathcal{L}(x, y) \quad (4.1)$$

under variations of the measure ρ , keeping the total volume $\rho(\mathcal{F})$ fixed (*volume constraint*). If $\rho(\mathcal{F})$ is infinite, it is not obvious how to implement the volume constraint, making it necessary to proceed as follows: First, we make the following additional assumptions:

- (iii) The measure ρ is *locally finite* (meaning that any $x \in \mathcal{F}$ has an open neighborhood U with $\rho(U) < \infty$).
- (iv) The function $\mathcal{L}(x, \cdot)$ is ρ -integrable for all $x \in \mathcal{F}$, giving a lower semi-continuous and bounded function on \mathcal{F} .

We remark that, since a manifold is second countable, property (iii) implies that ρ is σ -finite. In view of the computations later in this paper, it is most convenient to subtract a constant $\nu/2$ from the integral over $\mathcal{L}(x, \cdot)$ by introducing the function

$$\ell(x) = \int_{\mathcal{F}} \mathcal{L}(x, y) d\rho(y) - \frac{\nu}{2} : \mathcal{F} \rightarrow \mathbb{R} \quad \text{bounded and lower semi-continuous ,} \quad (4.2)$$

where the parameter $\nu \in \mathbb{R}$ will be specified below. We let $\tilde{\rho}$ be another Borel measure on \mathcal{F} which satisfies the conditions

$$|\tilde{\rho} - \rho|(\mathcal{F}) < \infty \quad \text{and} \quad (\tilde{\rho} - \rho)(\mathcal{F}) = 0 \quad (4.3)$$

(where $|\cdot|$ denotes the total variation of a measure; see [Hal74, §28] or [Rud87, Section 6.1]). Then the difference of the actions as given by

$$\begin{aligned} (\mathcal{S}(\tilde{\rho}) - \mathcal{S}(\rho)) &= \int_{\mathcal{F}} d(\tilde{\rho} - \rho)(x) \int_{\mathcal{F}} d\rho(y) \mathcal{L}(x, y) \\ &\quad + \int_{\mathcal{F}} d\rho(x) \int_{\mathcal{F}} d(\tilde{\rho} - \rho)(y) \mathcal{L}(x, y) + \int_{\mathcal{F}} d(\tilde{\rho} - \rho)(x) \int_{\mathcal{F}} d(\tilde{\rho} - \rho)(y) \mathcal{L}(x, y) \end{aligned} \quad (4.4)$$

is well-defined in view of the following lemma.

Lemma 4.1.1. *The integrals in (4.4) are well-defined with values in $\mathbb{R} \cup \{\infty\}$. Moreover,*

$$\begin{aligned} (\mathcal{S}(\tilde{\rho}) - \mathcal{S}(\rho)) &= 2 \int_{\mathcal{F}} \left(\ell(x) + \frac{\nu}{2} \right) d(\tilde{\rho} - \rho)(x) \\ &\quad + \int_{\mathcal{F}} d(\tilde{\rho} - \rho)(x) \int_{\mathcal{F}} d(\tilde{\rho} - \rho)(y) \mathcal{L}(x, y) . \end{aligned} \quad (4.5)$$

Proof. Decomposing the signed measure $\mu = \tilde{\rho} - \rho$ into its positive and negative parts, $\mu = \mu^+ - \mu^-$ (see the Jordan decomposition in [Hal74, §29]), the measures μ^\pm are both positive measures of finite total volume and $\mu^- \leq \rho$. In order to show that the integrals in (4.4) are well-defined, we need to prove that the negative contributions are finite, i.e.

$$\int_{\mathcal{F}} d\mu^-(x) \int_{\mathcal{F}} d\rho \mathcal{L}(x, y) < \infty \quad \text{and} \quad \int_{\mathcal{F}} d\mu^+(x) \int_{\mathcal{F}} d\mu^-(y) \mathcal{L}(x, y) < \infty \quad (4.6)$$

(here we apply Tonelli's theorem and make essential use of the fact that the Lagrangian is non-negative). The bounds (4.6) follow immediately from the estimates

$$\begin{aligned} \int_{\mathcal{F}} d\mu^-(x) \int_{\mathcal{F}} d\rho \mathcal{L}(x, y) &= \int_{\mathcal{F}} d\mu^-(x) \left(\ell(x) + \frac{\nu}{2} \right) \leq \left(\sup_{\mathcal{F}} \ell + \frac{\nu}{2} \right) \mu^-(\mathcal{F}) < \infty \\ \int_{\mathcal{F}} d\mu^+(x) \int_{\mathcal{F}} d\mu^-(y) \mathcal{L}(x, y) &\leq \int_{\mathcal{F}} d\mu^+(x) \int_{\mathcal{F}} d\rho \mathcal{L}(x, y) \\ &= \int_{\mathcal{F}} d\mu^+(x) \left(\ell(x) + \frac{\nu}{2} \right) \leq \left(\sup_{\mathcal{F}} \ell + \frac{\nu}{2} \right) \mu^+(\mathcal{F}) < \infty , \end{aligned}$$

where we used the fact that ℓ is assumed to be a bounded function on \mathcal{F} . \square

Definition 4.1.2. *The measure ρ is said to be a **minimizer** of the causal action if the difference (4.5) is non-negative for all $\tilde{\rho}$ satisfying (4.3),*

$$(\mathcal{S}(\tilde{\rho}) - \mathcal{S}(\rho)) \geq 0 .$$

We close this section with a remark on the existence theory. If \mathcal{F} is compact, the existence of minimizers can be shown just as in [Fin10a, Section 1.2] using the Banach-Alaoglu theorem (the fact that \mathcal{L} is semi-continuous implies that the weak-*limit of a minimizing sequence of measures is indeed a minimizer). In the non-compact setting, the existence theory has not yet been developed (for more details on this point see [Fin16b, §1.1.1]). For the purpose of the present paper, all we need is that the causal action principle admits *local* minimizers which satisfy the corresponding Euler-Lagrange equations. These concepts will be introduced in Sections 4.1.2 and 4.1.3 below. Moreover, in Section 4.4 we will analyze an example where local minimizers exist although \mathcal{F} is non-compact.

4.1.2 The Euler-Lagrange Equations

We now derive the Euler-Lagrange (EL) equations, following the method in the compact setting [FS13, Lemma 3.4].

Lemma 4.1.3. (Euler-Lagrange equations) *Let ρ be a minimizer of the causal action. Then*

$$\ell|_{\text{supp } \rho} \equiv \inf_{\mathcal{F}} \ell . \quad (4.7)$$

Proof. Given $x_0 \in \text{supp } \rho$, we choose an open neighborhood U with $0 < \rho(U) < \infty$. For any $y \in \mathcal{F}$ we consider the family of measures $(\tilde{\rho}_\tau)_{\tau \in [0,1]}$ given by

$$\tilde{\rho}_\tau = \chi_{M \setminus U} \rho + (1 - \tau) \chi_U \rho + \tau \rho(U) \delta_y$$

(where δ_y is the Dirac measure supported at y). Then

$$\tilde{\rho}_\tau - \rho = -\tau \chi_U \rho + \tau \rho(U) \delta_y = \tau (\rho(U) \delta_y - \chi_U \rho) , \quad (4.8)$$

implying that $\tilde{\rho}_\tau$ satisfies (4.3). Hence

$$0 \leq (\mathcal{S}(\tilde{\rho}) - \mathcal{S}(\rho)) = 2\tau \left(\rho(U) \left(\ell(y) + \frac{\nu}{2} \right) - \int_U \left(\ell(x) + \frac{\nu}{2} \right) d\rho(x) \right) + \mathcal{O}(\tau^2) .$$

As a consequence, the linear term must be non-negative,

$$\ell(y) \geq \frac{1}{\rho(U)} \int_U \ell(x) d\rho(x) . \quad (4.9)$$

Assume that (4.7) is false. Then there is $x_0 \in \text{supp } \rho$ and $y \in \mathcal{F}$ such that $\ell(x_0) > \ell(y)$. Lower semi-continuity of ℓ implies that there is an open neighborhood U of x_0 such that $\ell(x) > \ell(y)$ for all $x \in U$, in contradiction to (4.9). This gives the result. \square

We always choose ν such that $\inf_{\mathcal{F}} \ell = 0$. Then the EL equations (4.7) become

$$\ell|_{\text{supp } \rho} \equiv \inf_{\mathcal{F}} \ell = 0 . \quad (4.10)$$

We remark that ν can be understood as the Lagrange multiplier describing the volume constraint; see [Fin16b, §1.4.1].

4.1.3 Local Minimizers and Second Variations

We now introduce the concept of local minimizers of causal variational principles and explore the connection to second variations. We derive a convenient criterion which ensures that a measure ρ is a local minimizer (Proposition 4.1.7). This criterion will be used in the example of Section 4.4 to prove the existence of local minimizers.

We again consider families of variations $(\tilde{\rho}_\tau)_{\tau \in [0,\delta]}$ with $\tilde{\rho}_0 = \rho$ and assume that the measures $\tilde{\rho}_\tau$ all satisfy the conditions in (4.3). Then the family of measures μ_τ defined by

$$\mu_\tau := \tilde{\rho}_\tau - \rho , \quad (4.11)$$

are in the Banach space $\mathfrak{B}(\mathcal{F})$ of signed measures on \mathcal{F} with the norm given by the total variation.

Definition 4.1.4. *The measure ρ is a **local minimizer** of the causal action if for every family $(\tilde{\rho}_\tau)_{\tau \in [0, \delta)}$ of Borel measures which has the property that μ_τ defined by (4.11) is a smooth regular curve $\mu : [0, \delta) \rightarrow \mathfrak{B}(\mathcal{F})$ with $\mu_\tau(\mathcal{F}) = 0$, there is $\delta_0 \in (0, \delta)$ such that*

$$(\mathcal{S}(\tilde{\rho}_\tau) - \mathcal{S}(\rho)) \geq 0 \quad \text{for all } \tau \in [0, \delta_0). \quad (4.12)$$

We first derive the implications of local minimality. To this end, we assume that ρ is a local minimizer. Then obviously the EL equations (4.7) hold, because the curve (4.8) has the properties in the above definition. We consider variations $(\tilde{\rho}_\tau)_{\tau \in (-\delta, \delta)}$ of the form

$$\tilde{\rho}_\tau = (1 + \tau\psi) \rho, \quad (4.13)$$

where ψ is a real-valued function on \mathcal{F} . In order to ensure that these measures are again positive for sufficiently small $\delta > 0$, we must assume that ψ is an essentially bounded function. Moreover, these measures satisfy the conditions in the above definition if and only if

$$\int_M |\psi| d\rho < \infty \quad \text{and} \quad \int_M \psi d\rho = 0.$$

Hence we must assume that ψ is in the space

$$\mathcal{D}(\mathcal{L}_\rho) := \left\{ \psi \in (L^1 \cap L^\infty)(M, d\rho) \mid \int_M \psi d\rho = 0 \right\}. \quad (4.14)$$

By interpolation, the function ψ is also a vector in the Hilbert space $L^2(M, d\rho)$, also denoted by $(\mathcal{H}_\rho, \langle \cdot, \cdot \rangle_\rho)$. The operator \mathcal{L}_ρ in the following lemma was already analyzed in [FS13, Lemma 3.5] in the compact setting. We now extend this analysis to the non-compact setting.

Lemma 4.1.5. *For $\psi \in (L^1 \cap L^\infty)(M, d\rho)$, the function $\mathcal{L}_\rho \psi$ defined by*

$$(\mathcal{L}_\rho \psi)(x) = \int_M \mathcal{L}(x, y) \psi(y) d\rho(y)$$

is in $L^2(M, d\rho)$, giving rise to a linear operator

$$\mathcal{L}_\rho : \mathcal{D}(\mathcal{L}_\rho) \subset \mathcal{H}_\rho \rightarrow \mathcal{H}_\rho.$$

Proof. We apply Tonelli's theorem to obtain

$$\begin{aligned}
 \int_M |\mathcal{L}_\rho \psi|^2 d\rho &\leq \|\psi\|_{L^\infty(M)} \int_M d\rho(x) \int_M d\rho(y) \mathcal{L}(x, y) |\psi(y)| \int_M d\rho(y') \mathcal{L}(x, y') \\
 &= \|\psi\|_{L^\infty(M)} \int_M d\rho(x) \int_M d\rho(y) \mathcal{L}(x, y) |\psi(y)| \left(\ell(x) + \frac{\nu}{2} \right) \\
 &\leq \|\psi\|_{L^\infty(M)} \sup_M \left(\ell + \frac{\nu}{2} \right) \int_M d\rho(x) \int_M d\rho(y) \mathcal{L}(x, y) |\psi(y)| \\
 &= \|\psi\|_{L^\infty(M)} \sup_M \left(\ell + \frac{\nu}{2} \right) \int_M d\rho(y) |\psi(y)| \int_M d\rho(x) \mathcal{L}(x, y) \\
 &\leq \|\psi\|_{L^\infty(M)} \sup_M \left(\ell + \frac{\nu}{2} \right)^2 \|\psi\|_{L^1(M)} < \infty,
 \end{aligned}$$

where we used that \mathcal{L} is symmetric and that ℓ is bounded according to our assumption (4.2). This gives the result. \square

Proposition 4.1.6. *If ρ is a local minimizer, then the operator $\mathcal{L}_\rho : \mathcal{D}(\mathcal{L}_\rho) \rightarrow \mathcal{H}_\rho$ is positive (but not necessarily strictly positive).*

Proof. Computing (4.5) for the variation (4.13) gives

$$\begin{aligned}
 (\mathcal{S}(\tilde{\rho}) - \mathcal{S}(\rho)) &= 2\tau \int_{\mathcal{F}} \left(\ell(x) + \frac{\nu}{2} \right) \psi(x) d\rho \\
 &\quad + \tau^2 \int_{\mathcal{F}} \psi(x) d\rho(x) \int_{\mathcal{F}} \psi(y) d\rho(y) \mathcal{L}(x, y).
 \end{aligned} \tag{4.15}$$

The first summand vanishes in view of the EL equations (4.7). The second summand, on the other hand, exists in view of the estimates

$$\begin{aligned}
 \int_{\mathcal{F}} \psi(x) d\rho(x) \int_{\mathcal{F}} \psi(y) d\rho(y) \mathcal{L}(x, y) &\leq \|\psi\|_{L^\infty(M)} \int_{\mathcal{F}} \psi(x) d\rho(x) \int_{\mathcal{F}} d\rho(y) \mathcal{L}(x, y) \\
 &\leq \|\psi\|_{L^\infty(M)} \sup_M \left(\ell + \frac{\nu}{2} \right) \int_{\mathcal{F}} \psi(x) d\rho(x) = \|\psi\|_{L^\infty(M)} \sup_M \left(\ell + \frac{\nu}{2} \right) \|\psi\|_{L^1(M)}.
 \end{aligned}$$

Rewriting the second summand in (4.15) as an expectation value, we obtain

$$(\mathcal{S}(\tilde{\rho}) - \mathcal{S}(\rho)) = \tau^2 \langle \psi, \mathcal{L}_\rho \psi \rangle_\rho.$$

Applying the inequality (4.12) gives the result. \square

We finally give a criterion which ensures that ρ is a local minimizer.

Proposition 4.1.7. *Let ρ be a Borel measure with the following properties:*

(a) *The EL equations (4.10) are satisfied and in addition*

$$\ell(x) = 0 \implies x \in \text{supp } \rho. \tag{4.16}$$

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- (b) The Lagrangian $\mathcal{L} : \mathcal{F} \times \mathcal{F} \rightarrow \mathbb{R}_0^+$ is a bounded function.
- (c) The operator $\mathcal{L}_\rho : \mathcal{D}(\mathcal{L}_\rho) \rightarrow \mathcal{H}_\rho$ is strictly positive in the sense that there is $\varepsilon > 0$ such that

$$\langle \psi, \mathcal{L}_\rho \psi \rangle_\rho \geq \varepsilon \|\psi\|_\rho^2 \quad \text{for all } \psi \in \mathcal{D}(\mathcal{L}_\rho). \quad (4.17)$$

Then ρ is a local minimizer.

We remark that condition (b) could be replaced by weaker boundedness assumptions. We do not aim for maximal generality because condition (b) is suitable for the applications we have in mind.

Proof of Proposition 4.1.7. Let $(\tilde{\rho}_\tau)_{\tau \in [0, \delta]}$ be as in Definition 4.1.4. Since the curve μ_τ in Definition 4.1.4 is regular, we know that $\dot{\tilde{\rho}}_0$ is non-zero. Expanding (4.5) in powers of τ , we obtain

$$\begin{aligned} (\mathcal{S}(\tilde{\rho}) - \mathcal{S}(\rho)) &= 2\tau \int_{\mathcal{F}} \left(\ell(x) + \frac{\nu}{2} \right) d\dot{\tilde{\rho}}_0(x) + \tau^2 \int_{\mathcal{F}} \left(\ell(x) + \frac{\nu}{2} \right) d\ddot{\tilde{\rho}}_0(x) \\ &\quad + 2\tau^2 \int_{\mathcal{F}} d\dot{\tilde{\rho}}_0(x) \int_{\mathcal{F}} d\dot{\tilde{\rho}}_0(x) \mathcal{L}(x, y) + \mathcal{O}(\tau^3). \end{aligned}$$

Due to the volume constraint, the signed measures $\dot{\tilde{\rho}}_0$ and $\ddot{\tilde{\rho}}_0$ have total volume zero, so that the terms involving ν drop out,

$$\begin{aligned} (\mathcal{S}(\tilde{\rho}) - \mathcal{S}(\rho)) &= 2\tau \int_{\mathcal{F}} \ell(x) d\dot{\tilde{\rho}}_0(x) + \tau^2 \int_{\mathcal{F}} \ell(x) d\ddot{\tilde{\rho}}_0(x) \\ &\quad + 2\tau^2 \int_{\mathcal{F}} d\dot{\tilde{\rho}}_0(x) \int_{\mathcal{F}} d\dot{\tilde{\rho}}_0(x) \mathcal{L}(x, y) + \mathcal{O}(\tau^3). \end{aligned} \quad (4.18)$$

We first consider the case that the measure $\chi_{\mathcal{F} \setminus M} \dot{\tilde{\rho}}_0$ is non-zero. Since the measures $\tilde{\rho}_\tau$ are all positive, we know that $\chi_{\mathcal{F} \setminus M} \dot{\tilde{\rho}}_0$ is a positive measure. Hence, using (4.16), we conclude that

$$\int_{\mathcal{F}} \ell(x) d\dot{\tilde{\rho}}_0(x) > 0.$$

Hence the linear term in (4.18) ensures that (4.12) holds for sufficiently small τ .

It remains to consider the case that the measure $\dot{\tilde{\rho}}_0$ is supported on M . Then the linear term in (4.18) vanishes because of the EL equations (4.7). Repeating the above argument with $\dot{\tilde{\rho}}_0$ replaced by $\ddot{\tilde{\rho}}_0$, we find that $\chi_{\mathcal{F} \setminus M} \ddot{\tilde{\rho}}_0$ is a positive measure. From this it follows that

$$\int_{\mathcal{F}} \ell(x) d\ddot{\tilde{\rho}}_0(x) \geq 0.$$

Therefore, in order to conclude the proof, it remains to show that

$$\int_M d\dot{\tilde{\rho}}_0(x) \int_M d\dot{\tilde{\rho}}_0(y) \mathcal{L}(x, y) > 0. \quad (4.19)$$

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We now use the following approximation argument. We choose a sequence $\psi_n \in \mathcal{D}(\mathcal{L}_\rho)$ such that

$$\psi_n \rho \rightarrow \dot{\rho}_0 \neq 0 \quad \text{in } \mathfrak{B}(\mathcal{F}). \quad (4.20)$$

Then

$$\int_M \psi_n(x) d\rho(x) \int_M \psi_n(y) d\rho(y) \mathcal{L}(x, y) \rightarrow \int_M d\dot{\rho}_0(x) \int_M d\dot{\rho}_0(y) \mathcal{L}(x, y),$$

because, setting $\dot{\rho}_0 = \psi_n \rho + \Delta\rho$, we have

$$\begin{aligned} & \int_M d\dot{\rho}_0(x) \int_M d\dot{\rho}_0(y) \mathcal{L}(x, y) - \int_M \psi_n(x) d\rho(x) \int_M \psi_n(y) d\rho(y) \mathcal{L}(x, y) \\ &= \int_M d\Delta\rho(x) \int_M d\Delta\rho(y) \mathcal{L}(x, y) + 2 \int_M d\Delta\rho(x) \int_M \psi_n(y) d\rho(y) \mathcal{L}(x, y) \\ &\leq C \|\Delta\rho\|_{\mathfrak{B}(\mathcal{F})}^2 + 2 \|\psi_n\|_{L^\infty(M)} \sup_M \left(\ell + \frac{\nu}{2} \right) \|\Delta\rho\|_{\mathfrak{B}(\mathcal{F})} \rightarrow 0, \end{aligned}$$

where $C := \sup_{x, y \in \mathcal{F}} \mathcal{L}(x, y)$ is the pointwise bound of the Lagrangian. Using the strict positivity (4.17), we have

$$\varepsilon \|\psi_n\|_\rho^2 \leq \langle \psi_n, \mathcal{L}_\rho \psi_n \rangle_{\mathcal{H}} = \int_M \psi_n(x) d\rho(x) \int_M \psi_n(y) d\rho(y) \mathcal{L}(x, y), \quad (4.21)$$

hence the left hand side of (4.19) cannot be negative. Let us assume that it is zero,

$$\int_M d\dot{\rho}_0(x) \int_M d\dot{\rho}_0(y) \mathcal{L}(x, y) = 0. \quad (4.22)$$

Using (4.21), it follows that

$$\|\psi_n\|_\rho^2 \rightarrow 0.$$

Thus $\psi_n \rightarrow 0$ converges pointwise almost everywhere in M . It follows that $\psi_n \rho \rightarrow 0$ in $\mathfrak{B}(\mathcal{F})$, in contradiction to (4.20). This shows that assumption (4.22) is false, concluding the proof. \square

4.1.4 The Setting of Causal Fermion Systems

We now explain how the causal action principle for causal fermion systems (as introduced in Section 3.3.1) can be described within the above setting. The main difference compared to the setting in Section 4.1.1 is that the causal action principle involves additional constraints, namely the trace constraint and the boundedness constraint. We now explain how to incorporate these constraints in a convenient way. For a minimizer of the causal action, the local trace is constant on the support of the universal measure (see [Fin16b, Proposition 1.4.1]). With this in mind, we may restrict attention to operators with fixed trace. When doing so, the trace constraint is trivially satisfied. The boundedness

constraint, on the other hand, can be incorporated by Lagrange multiplier term. Finally, in the setting of causal fermion systems, the set \mathcal{F} is not necessarily a smooth manifold. In order to avoid this problem, we restrict attention to minimizers for which all space-time points are regular (see [Fin16b, Definition 1.1.5]). Then we may restrict attention to operators which have exactly n positive and n negative eigenvalues. The resulting set of operators is a smooth manifold (see the concept of a flag manifold in [Hel00]). This leads us to the following setup:

Let $(\mathcal{H}, \langle \cdot, \cdot \rangle_{\mathcal{H}})$ be a finite-dimensional complex Hilbert space. Moreover, we are given parameters $n \in \mathbb{N}$ (the spin dimension), $c > 0$ (the constraint for the local trace) and $\kappa > 0$ (the Lagrange multiplier of the boundedness constraint)¹. We let $\mathcal{F} \subset L(\mathcal{H})$ be the set of all self-adjoint operators F on \mathcal{H} with the following properties:

- F has finite rank and (counting multiplicities) has n positive and n negative eigenvalues.
- The local trace is constant, i.e.

$$\mathrm{tr}(F) = c.$$

On \mathcal{F} we consider the topology induced by the sup-norm on $L(\mathcal{H})$. For any $x, y \in \mathcal{F}$, the product xy is an operator of rank at most $2n$. We denote its non-trivial eigenvalues counting algebraic multiplicities by $\lambda_1^{xy}, \dots, \lambda_{2n}^{xy} \in \mathbb{C}$. We introduce the Lagrangian by

$$\mathcal{L}(x, y) = \frac{1}{4n} \sum_{i,j=1}^{2n} \left(|\lambda_i^{xy}| - |\lambda_j^{xy}| \right)^2 + \kappa \left(\sum_{i,j=1}^{2n} |\lambda_i^{xy}| \right)^2. \quad (4.23)$$

Clearly, this Lagrangian is continuous on $\mathcal{F} \times \mathcal{F}$. Moreover, since $\kappa > 0$, the Lagrangian is non-negative.

Therefore, we are back in the setting of Section 4.1.1. The EL equations in Lemma 4.1.3 agree with the EL equations as derived for the causal action principle with constraints in [BF14] (cf. Theorem 3.3.2).

4.2 The Symplectic Form in the Smooth Setting

In order to introduce our concepts in the simplest possible setting, in this section we assume that \mathcal{F} is a smooth manifold of dimension $m \geq 1$ and that the Lagrangian $\mathcal{L} \in C^\infty(\mathcal{F} \times \mathcal{F}, \mathbb{R}_0^+)$ is smooth. Moreover, we let ρ be a regular Borel measure on \mathcal{F} which satisfies the EL equations (4.10) corresponding to the causal action in the sense that the smooth function ℓ defined by (4.2),

$$\ell(x) = \int_{\mathcal{F}} \mathcal{L}(x, y) d\rho(y) - \frac{\nu}{2} \in C^\infty(\mathcal{F}, \mathbb{R}_0^+),$$

¹We remark that the Lagrange multiplier κ is strictly positive because otherwise there are no minimizers; see [Fin10a, Example 2.9] and [Fin16b, Exercise 1.4].

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is minimal and vanishes on M ,

$$\ell|_{\text{supp } \rho} \equiv \inf_{\mathcal{F}} \ell = 0. \quad (4.24)$$

The constructions in this section should be seen as a preparation for the lower semi-continuous setting to be considered in Section 4.3. Before going on, we remark that the value of the parameter ν can be changed arbitrarily by rescaling the measure according to

$$\rho \rightarrow \lambda \rho \quad \text{with} \quad \lambda > 0.$$

Therefore, without loss of generality we can keep ν fixed when varying or perturbing the measure.

4.2.1 The Weak Euler-Lagrange Equations

Clearly, the EL equations (4.24) imply the weaker equations

$$\ell|_M \equiv 0 \quad \text{and} \quad D\ell|_M \equiv 0 \quad (4.25)$$

(where $D\ell(p) : T_p\mathcal{F} \rightarrow \mathbb{R}$ is the derivative). In order to combine these two equations in a compact form, we introduce the smooth one-jets

$$\mathfrak{J} := \{\mathbf{u} = (a, u) \text{ with } a \in C^\infty(\mathcal{F}, \mathbb{R}) \text{ and } u \in \Gamma(\mathcal{F})\}, \quad (4.26)$$

where $\Gamma(\mathcal{F})$ denotes the smooth vector fields on \mathcal{F} . Defining the derivative in direction of a one-jet by

$$\nabla_{\mathbf{u}}\ell(x) := a(x)\ell(x) + (D_u\ell)(x), \quad (4.27)$$

we can write (4.25) as

$$\nabla_{\mathbf{u}}\ell|_M \equiv 0 \quad \text{for all } \mathbf{u} \in \mathfrak{J}. \quad (4.28)$$

We refer to these equations as the *weak EL equations*.

4.2.2 The Nonlinear Solution Space

Our next step is to analyze families of measures which satisfy the weak EL equations. In order to obtain these families of solutions, we want to vary a given measure ρ_0 (not necessarily a minimizer) without changing its general structure. To this end, we multiply ρ_0 by a weight function and apply a diffeomorphism, i.e.

$$\rho = F_*(f\rho_0), \quad (4.29)$$

where $F : \mathcal{F} \rightarrow \mathcal{F}$ is a smooth diffeomorphism and $f \in C^\infty(\mathcal{F}, \mathbb{R}^+)$. We now consider a set of such measures which all satisfy the weak EL equations,

$$\mathcal{B} \subset \{\rho \text{ of the form (4.29)} \mid \text{the weak EL equations (4.28) are satisfied}\} \quad (4.30)$$

(for fixed ρ_0). We make further simplifying assumptions on \mathcal{B} . First, we assume that \mathcal{B} is a smooth *Fréchet manifold* (endowed with the compact-open topology on $C^\infty(\mathcal{F}, \mathbb{R}_0^+)$ and on the diffeomorphisms; for details see Appendix A). Then for $\rho \in \mathcal{B}$, a tangent vector $\mathbf{v} \in T_\rho \mathcal{B}$, being an infinitesimal variation of the measures in (4.29), consists of a function b (describing the infinitesimal change of the weight) and a vector field v (being the infinitesimal generator of the diffeomorphism). Having chosen the Fréchet topology such that b and v are smooth, we obtain a jet $\mathbf{v} = (b, v) \in \mathfrak{J}$. Hence the tangent space can be identified with a subspace of the one-jets,

$$\mathcal{T}_\rho \mathcal{B} \subset \mathfrak{J}.$$

A second assumption is needed in order to ensure that we can exchange integration with differentiation in the proof of Lemma 4.2.1 below. To this end, we assume that for every smooth curve $(\tilde{\rho}_\tau)_{\tau \in (-\delta, \delta)}$ in \mathcal{B} , the corresponding functions (f_τ, F_τ) have the properties that the derivatives

$$\frac{d}{d\tau} \mathcal{L}(F_\tau(x), F_\tau(y)) f_\tau(y) \quad \text{and} \quad \frac{d}{d\tau} D_1 \mathcal{L}(F_\tau(x), F_\tau(y)) f_\tau(y) \quad (4.31)$$

are bounded uniformly in τ for every $x, y \in M$ and are ρ -integrable in y for every $x \in M$ (just as in (1.7), the subscripts of D_1 and D_2 denote the partial derivatives acting on the first respectively second argument of the Lagrangian).

Lemma 4.2.1. *For any $\mathbf{u} \in \mathfrak{J}$ and $\mathbf{v} \in T_\rho \mathcal{B}$,*

$$\nabla_{\mathbf{u}} \nabla_{\mathbf{v}} \ell(x) + \int_M \nabla_{1, \mathbf{u}} \nabla_{2, \mathbf{v}} \mathcal{L}(x, y) d\rho(y) = 0 \quad \text{for all } x \in M. \quad (4.32)$$

We refer to (4.32) as the *linearized field equations* (see also the explanation in the introduction before (1.7)).

Before giving the proof of this lemma, we point out that in (4.32), the order of differentiation is irrelevant. This is obvious for the term $\nabla_{1, \mathbf{u}} \nabla_{2, \mathbf{v}} \mathcal{L}(x, y)$ because the derivatives act on different variables. In the first term, it follows from the computation

$$\nabla_{\mathbf{u}} \nabla_{\mathbf{v}} \ell(x) - \nabla_{\mathbf{v}} \nabla_{\mathbf{u}} \ell(x) = (D_{\mathbf{u}} a)(x) \ell(x) - (D_{\mathbf{v}} b)(x) \ell(x) + D_{[\mathbf{u}, \mathbf{v}]} \ell(x) = 0,$$

where in the last step we used the weak EL equations (4.25).

Proof of Lemma 4.2.1. Given $\mathbf{v} = (b, v) \in T_\rho \mathcal{B}$, we let $(\tilde{\rho}_\tau)_{\tau \in (-\delta, \delta)}$ be a smooth curve in \mathcal{B} with $\tilde{\rho}_0 = \rho$ and $\dot{\tilde{\rho}}_0 = \mathbf{v}$. As shown in Lemma 4.2.2 below, there are F_τ and f_τ such that

$$\tilde{\rho}_\tau = (F_\tau)_* (f_\tau \rho), \quad (4.33)$$

and therefore

$$\dot{\tilde{\rho}}_0 = \frac{d}{d\tau} \left((F_\tau)_* (f_\tau \rho) \right) \Big|_{\tau=0} \quad \text{with} \quad \dot{f}_0 = b, \quad \dot{F}_0 = v. \quad (4.34)$$

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Setting $M_\tau = \text{supp } \tilde{\rho}_\tau$ and using that $M_\tau = \overline{F_\tau(M)}$, the weak EL equations (4.25) can be written as

$$\ell_\tau(F_\tau(x)) \equiv 0 \quad \text{and} \quad D\ell_\tau(F_\tau(x)) \equiv 0 \quad \text{for all } x \in M, \quad (4.35)$$

where

$$\ell_\tau(z) := \int_{\mathcal{F}} \mathcal{L}(z, y) d\tilde{\rho}_\tau(y) - \frac{\nu}{2} \in C^\infty(\mathcal{F}, \mathbb{R}). \quad (4.36)$$

Differentiating the first equation in (4.35) with respect to τ , we obtain

$$\begin{aligned} 0 &= \frac{d}{d\tau} \ell_\tau(F_\tau(x)) \Big|_{\tau=0} = \frac{d}{d\tau} \int_{\mathcal{F}} \mathcal{L}(F_\tau(x), y) d\left((F_\tau)_*(f_\tau \rho)\right)(y) \Big|_{\tau=0} \\ &= \frac{d}{d\tau} \int_M \mathcal{L}(F_\tau(x), F_\tau(y)) f_\tau(y) d\rho(y) \Big|_{\tau=0} \\ &= D_v \ell(x) + \int_M \mathcal{L}(x, y) b(y) d\rho(y) + \int_{\mathcal{F}} D_{2,v} \mathcal{L}(x, y) d\rho(y). \end{aligned}$$

In the last step, we exchanged integration with differentiation. This is justified by our assumption (4.31), which ensures that the integrand of the second line is $L^1(\mathcal{F}, d\rho)$ for every τ , is differentiable in τ for every $x, y \in M$ and is dominated by a $L^1(\mathcal{F}, \rho)$ -function uniformly in τ . Using the notation (4.27), we can write this as

$$D_v \ell(x) + \int_M \nabla_{2,v} \mathcal{L}(x, y) d\rho(y) = 0. \quad (4.37)$$

Differentiating the second equation in (4.35), a similar computation gives for any vector field u

$$D_v D_u \ell(x) + \int_M D_{1,u} \nabla_{2,v} \mathcal{L}(x, y) d\rho(y) = 0. \quad (4.38)$$

Multiplying (4.37) by $a(x)$ and adding (4.38), we obtain

$$\begin{aligned} 0 &= a(x) D_v \ell(x) + D_v D_u \ell(x) + \int_M \nabla_{1,u} \nabla_{2,v} \mathcal{L}(x, y) d\rho(y) \\ &= D_v \nabla_u \ell(x) - (D_v a)(x) \ell(x) + \int_M \nabla_{1,u} \nabla_{2,v} \mathcal{L}(x, y) d\rho(y) \\ &= \nabla_v \nabla_u \ell(x) - b(x) \nabla_u \ell(x) - (D_v a)(x) \ell(x) + \int_M \nabla_{1,u} \nabla_{2,v} \mathcal{L}(x, y) d\rho(y). \end{aligned}$$

Using the weak EL equations (4.28), the second and third summands vanish, giving the result. \square

Lemma 4.2.2. *Let $\rho \in \mathcal{B}$ and $(\tilde{\rho}_\tau)_{\tau \in (-\delta, \delta)}$ be a curve such that $\tilde{\rho}_0 = \rho$. Then there is a*

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family of smooth diffeomorphisms $F_\tau : \mathcal{F} \rightarrow \mathcal{F}$ and functions $f_\tau \in C^\infty(\mathcal{F}, \mathbb{R}^+)$ such that

$$\tilde{\rho}_\tau = (F_\tau)_*(f_\tau \rho). \quad (4.39)$$

If the curve $(\tilde{\rho}_\tau)_{\tau \in (-\delta, \delta)}$ is smooth, then both F_τ and f_τ are smooth in τ .

Proof. Let ρ_0 be the measure in the definition of \mathcal{B} , (4.30). Then there are $G : \mathcal{F} \rightarrow \mathcal{F}$ and $g \in C^\infty(\mathcal{F}, \mathbb{R}^+)$ such that

$$\rho = G_*(g \rho_0).$$

Thus

$$\rho_0 = \frac{1}{g} \left((G^{-1})_* \rho \right) = (G^{-1})_* \left(\frac{1}{g \circ G^{-1}} \rho \right).$$

Similarly, there are mappings $\tilde{G} : \mathcal{F} \rightarrow \mathcal{F}$ and $\tilde{g} \in C^\infty(\mathcal{F}, \mathbb{R}^+)$ such that

$$\begin{aligned} \tilde{\rho}_\tau &= \tilde{G}_*(\tilde{g} \rho_0) = \tilde{G}_* \left(\tilde{g} (G^{-1})_* \left(\frac{1}{g \circ G^{-1}} \rho \right) \right) \\ &= (\tilde{G} \circ G^{-1})_* \left(\frac{\tilde{g} \circ G^{-1}}{g \circ G^{-1}} \rho \right). \end{aligned}$$

This gives the desired functions F_τ and f_τ for fixed τ . The claim about smoothness follows from the topology and the differential structure of \mathcal{B} as described Appendix A (a curve (4.39) is smooth if and only if F_τ and f_τ are smooth in τ). \square

We remark that the strict positivity of the weight function f in (4.29) is needed because in the last proof we divided by these weight functions.

4.2.3 The Symplectic Form and Hamiltonian Time Evolution

For any $\mathbf{u}, \mathbf{v} \in T_\rho \mathcal{B}$ and $x, y \in M$, we set

$$\sigma_{\mathbf{u}, \mathbf{v}}(x, y) := \nabla_{1, \mathbf{u}} \nabla_{2, \mathbf{v}} \mathcal{L}(x, y) - \nabla_{1, \mathbf{v}} \nabla_{2, \mathbf{u}} \mathcal{L}(x, y).$$

For any compact $\Omega \subset \mathcal{F}$, we introduce the surface layer integral

$$\sigma_\Omega : T_\rho \mathcal{B} \times T_\rho \mathcal{B} \rightarrow \mathbb{R}, \quad \sigma_\Omega(\mathbf{u}, \mathbf{v}) = \int_\Omega d\rho(x) \int_{M \setminus \Omega} d\rho(y) \sigma_{\mathbf{u}, \mathbf{v}}(x, y). \quad (4.40)$$

We are now in the position to specify and prove the theorem mentioned in Section 1.2 of the introduction.

Theorem 4.2.3. *Let \mathcal{F} be a smooth manifold of dimension $m \geq 1$, and $\mathcal{L} \in C^\infty(\mathcal{F} \times \mathcal{F}, \mathbb{R}_0^+)$ be a smooth Lagrangian. Moreover, let \mathcal{B} be a Fréchet manifold of measures of the form (4.30). Then for any compact $\Omega \subset \mathcal{F}$, the surface layer integral (4.40) vanishes for all $\mathbf{u}, \mathbf{v} \in T_\rho \mathcal{B}$.*

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Proof. Anti-symmetrizing (4.32) in \mathbf{u} and \mathbf{v} and using that $\nabla_{[\mathbf{u}, \mathbf{v}]} \ell = 0$, we obtain

$$\int_M \sigma_{\mathbf{u}, \mathbf{v}}(x, y) d\rho(y) = 0 \quad \text{for all } x \in M.$$

We integrate this equation over Ω ,

$$\begin{aligned} 0 &= \int_{\Omega} d\rho(x) \int_M d\rho(y) \sigma_{\mathbf{u}, \mathbf{v}}(x, y) \\ &= \int_{\Omega} d\rho(x) \int_{\Omega} d\rho(y) \sigma_{\mathbf{u}, \mathbf{v}}(x, y) + \int_{\Omega} d\rho(x) \int_{M \setminus \Omega} d\rho(y) \sigma_{\mathbf{u}, \mathbf{v}}(x, y). \end{aligned} \quad (4.41)$$

Since the Lagrangian is symmetric in its two arguments, the function $\sigma_{\mathbf{u}, \mathbf{v}}$ is obviously anti-symmetric, i.e. $\sigma_{\mathbf{u}, \mathbf{v}}(x, y) = -\sigma_{\mathbf{u}, \mathbf{v}}(y, x)$. Therefore, the first summand in (4.41) vanishes. This gives the result. \square

At this point, we want to use the construction explained after (1.9) in the introduction to obtain a conserved symplectic form σ . In the present smooth setting, this construction can be made precise as follows. Let us assume that M is a smooth manifold being a topological product

$$M = \mathbb{R} \times N$$

with a (possibly non-compact) smooth manifold N . (This is for example the case if M is a globally hyperbolic Lorentzian manifold, where \mathbb{R} corresponds to the time of a foliation and N corresponds to space.) For any $t \in \mathbb{R}$, the set $N_t := \{t\} \times N$ is a hypersurface in M , and it can be realized as a boundary,

$$N_t = \partial\Omega_{N_t} \quad \text{with} \quad \Omega_{N_t} := (-\infty, 0) \times N.$$

Next, let us assume that the jets $\mathbf{v} = (b, v) \in T_{\rho}\mathcal{B}$ have suitable decay properties on N_t (at spatial infinity) which ensure that the surface layer integrals (4.40) exist for $\Omega = \Omega_{N_t}$ and every $t \in \mathbb{R}$. Under these assumptions, Theorem 4.2.3 implies that the bilinear form $\sigma_{\Omega_{N_t}}$ is well-defined and does not depend on t . This makes it possible to introduce the mapping

$$\sigma : T_{\rho}\mathcal{B} \times T_{\rho}\mathcal{B} \rightarrow \mathbb{R}, \quad (\mathbf{u}, \mathbf{v}) \mapsto \sigma_{\Omega_{N_t}}(\mathbf{u}, \mathbf{v}) \quad (4.42)$$

(where $t \in \mathbb{R}$ is arbitrary). Due to the anti-symmetry, we can regard σ as a two-form on \mathcal{B} . The next lemma shows that σ endows \mathcal{B} with the structure of a presymplectic Fréchet manifold.

Lemma 4.2.4. *The bilinear form σ is closed.*

Proof. Inspired by classical field theory (see for example [DF99, §2.3]), our strategy is to write σ locally as the exterior derivative of a one-form γ . Then the claim follows immediately from the fact that $d^2 = 0$.

We let $\tilde{\rho}$ be a measure in a neighborhood of $\rho \in \mathcal{B}$. By definition of \mathcal{B} we can represent $\tilde{\rho}$

as

$$\tilde{\rho} = F_*(f\rho) \in \mathcal{B}. \quad (4.43)$$

We next define $\gamma : T_{\tilde{\rho}}\mathcal{B} \rightarrow \mathbb{R}$ by

$$\gamma(\mathbf{u}) = \int_{\Omega_{N_t}} d\rho \int_{M \setminus \Omega_{N_t}} d\rho f(x) \nabla_{2,\mathbf{u}} \mathcal{L}(F(x), F(y)) f(y). \quad (4.44)$$

Computing the outer derivative with the formula

$$(d\gamma)(\mathbf{u}, \mathbf{v}) = \mathbf{u}\gamma(\mathbf{v}) - \mathbf{v}\gamma(\mathbf{u}) - \gamma([\mathbf{u}, \mathbf{v}]),$$

one finds that $\sigma = d\gamma$ (for details see Appendix A). This concludes the proof. \square

In order to obtain a symplectic structure on \mathcal{B} , the presymplectic form σ must be non-degenerate. We do not see a general reason why this should be the case. Therefore, we proceed as follows. Given $\rho \in \mathcal{B}$, an abstract method to obtain a non-degenerate form is to mod out the kernel of σ defined by

$$\ker \sigma = \{\mathbf{v} \in T_\rho \mathcal{B} \mid \sigma(\mathbf{u}, \mathbf{v}) = 0 \text{ for all } \mathbf{u} \in T_\rho \mathcal{B}\}.$$

In most applications, it is useful to choose concrete representatives of the vectors of this quotient space. To this end, one chooses a maximal subspace $\mathfrak{J}^{\text{symp}}$ of $T_\rho \mathcal{B}$ on which σ is non-degenerate (the existence of such a subspace is guaranteed by Zorn's lemma). Then the restriction

$$\sigma : \mathfrak{J}^{\text{symp}} \times \mathfrak{J}^{\text{symp}} \rightarrow \mathbb{R}$$

is non-degenerate. The specific choice of $\mathfrak{J}^{\text{symp}}$ depends on the application.

To summarize, the above constructions gave us a presymplectic form σ on $T_\rho \mathcal{B}$ which is given as a surface layer integral (1.8) for $\Omega = \Omega_{N_t}$. This presymplectic form is independent of t . In other words, the time evolution as specified by the linearized field equations (4.32) preserves the symplectic form and is thus a symplectomorphism. This is what we mean by *Hamiltonian time evolution*.

4.3 The Lower Semi-Continuous Setting

We now return to the lower semi-continuous setting of Section 4.1. We assume that ρ satisfies the EL equations of the causal action (see (4.2) and (4.10)). Thus we assume that the function $\ell : \mathcal{F} \rightarrow \mathbb{R}$ defined by

$$\ell(x) := \int_{\mathcal{F}} \mathcal{L}(x, y) d\rho(y) - \frac{\nu}{2} \quad \text{is bounded and lower semi-continuous,} \quad (4.45)$$

and that it is minimal on the support of ρ ,

$$\ell|_{\text{supp } \rho} \equiv \inf_{\mathcal{F}} \ell = 0 \quad (4.46)$$

(here $\nu > 0$ is again the Lagrange multiplier describing the volume constraint; see [Fin16b, §1.4.1]). We again introduce space-time as the support of the universal measure,

$$M := \text{supp } \rho . \quad (4.47)$$

4.3.1 The Weak Euler-Lagrange Equations

Since the function ℓ as defined in (4.45) is only lower semi-continuous, the derivative in (4.25) in general does not exist. But for lower semi-continuous functions, it is a reasonable assumption that the semi-derivatives exist, but may take the value $+\infty$. This leads us to the following additional assumptions:

- (v) \mathcal{L} has directional semi-derivatives in $\mathbb{R} \cup \{\infty\}$: For any $x, y \in \mathcal{F}$, $v \in T_x \mathcal{F}$ and any curve $\gamma \in C^1((-1, 1), \mathcal{F})$ with $\gamma(0) = x$ and $\gamma'(0) = v$, the following generalized semi-derivative exists

$$D_{1,v}^+ \mathcal{L}(x, y) := \lim_{\tau \searrow 0} \frac{1}{\tau} \left(\mathcal{L}(\gamma(\tau), y) - \mathcal{L}(\gamma(0), y) \right) \in \mathbb{R} \cup \{\infty\} \quad (4.48)$$

and is independent of the choice of γ .

- (vi) For any $x \in M$ and $v \in T_x \mathcal{F}$, both sides of the following equation exist and are equal,

$$D_v^+ \ell(x) = \int_M D_{1,v}^+ \mathcal{L}(x, y) d\rho(y) .$$

Under these assumptions, the EL equations (4.46) can again be tested weakly with smooth jets. We define the jet space \mathfrak{J} as in (4.26),

$$\mathfrak{J} := \{ \mathbf{u} = (a, u) \text{ with } a \in C^\infty(\mathcal{F}, \mathbb{R}) \text{ and } u \in C^\infty(\mathcal{F}, T\mathcal{F}) \} . \quad (4.49)$$

We remark that it would suffice to define the mappings f and F on an open neighborhood of M . But, keeping in mind that every such mapping can be extended smoothly to all of \mathcal{F} , there is no loss in generality to assume that f and F are defined on all of \mathcal{F} . When testing, only the restriction of the jets to M is of relevance. We thus define the jet space

$$\mathfrak{J}|_M := \{ \mathbf{u} = (a, u) \text{ with } a \in C^\infty(M, \mathbb{R}) \text{ and } u \in C^\infty(M, T\mathcal{F}) \} , \quad (4.50)$$

where smooth functions and sections on M are defined as those functions (respectively sections) which have a smooth extension to \mathcal{F} . Since in general only the semi-derivatives exist, in contrast to (4.28) the weak EL equations read

$$\nabla_{\mathbf{u}}^+ \ell(x) \geq 0 \quad \text{for all } x \in M \text{ and } \mathbf{u} \in \mathfrak{J}|_M , \quad (4.51)$$

where, similar to (4.27), $\nabla_{\mathbf{u}}^+$ is defined as

$$\nabla_{\mathbf{u}}^+ \ell(x) := a(x) \ell(x) + (D_u^+ \ell)(x) . \quad (4.52)$$

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We introduce $\mathfrak{J}^{\text{diff}}$ as the subspace of jets on M such that ℓ is differentiable in the direction of the vector field, i.e.

$$\mathfrak{J}^{\text{diff}} := \{\mathfrak{u} \in \mathfrak{J}|_M \text{ with } \nabla_{\mathfrak{u}}^+ \ell = -\nabla_{-\mathfrak{u}}^+ \ell\} \subset \mathfrak{J}|_M. \quad (4.53)$$

Note that the last equation does not impose a condition for the scalar component of the jet, so that

$$\begin{aligned} \mathfrak{J}^{\text{diff}} &= C^\infty(M, \mathbb{R}) \oplus \Gamma^{\text{diff}} \quad \text{where} \\ \Gamma^{\text{diff}} &:= \{u \in C^\infty(M, T\mathcal{F}) \text{ with } D_u^+ \ell = -D_{-u}^+ \ell\}. \end{aligned} \quad (4.54)$$

Thus for jets in $\mathfrak{J}^{\text{diff}}$, the directional derivatives exist, so that $\nabla_{\mathfrak{u}}^+ \ell = \nabla_u \ell$. Then (4.51) implies that

$$\nabla_{\mathfrak{u}} \ell(x) = 0 \quad \text{for all } x \in M \text{ and } \mathfrak{u} \in \mathfrak{J}^{\text{diff}}. \quad (4.55)$$

As explained in the introduction after (1.7), in physical applications it suffices to use only part of the information contained in these equations. To this end, we choose a linear subspace

$$\mathfrak{J}^{\text{test}} = C^{\text{test}}(M, \mathbb{R}) \oplus \Gamma^{\text{test}} \subset \mathfrak{J}^{\text{diff}} \quad (4.56)$$

and consider the *weak EL equations*

$$\nabla_{\mathfrak{u}} \ell|_M = 0 \quad \text{for all } \mathfrak{u} \in \mathfrak{J}^{\text{test}}. \quad (4.57)$$

The choice of $\mathfrak{J}^{\text{test}}$ depends on the specific application and is of no relevance for the remainder of this section.

4.3.2 Families of Solutions and Linearized Solutions

We now consider families of solutions of the weak EL equations. To this end, we let $(\tilde{\rho}_\tau)_{\tau \in (-\delta, \delta)}$ be a family of measures, which similar to (4.29) and (4.43) we assume to be of the form

$$\tilde{\rho}_\tau = (F_\tau)_*(f_\tau \rho), \quad (4.58)$$

where f and F are smooth,

$$f \in C^\infty((-\delta, \delta) \times \mathcal{F} \rightarrow \mathbb{R}^+) \quad \text{and} \quad F \in C^\infty((-\delta, \delta) \times \mathcal{F} \rightarrow \mathcal{F}),$$

and have the properties $f_0(x) = 1$ and $F_0(x) = x$ for all $x \in M$. Then the support of $\tilde{\rho}_\tau$ is given by

$$M_\tau := \text{supp } \tilde{\rho}_\tau = \overline{F_\tau(M)}.$$

In order to formulate the weak EL equations (4.57) for $\tau \neq 0$, there is the complication that the jets in $\mathfrak{J}^{\text{test}}$ are defined only on M , whereas the weak EL equations must be evaluated on M_τ . Therefore, we must introduce a jet space $\mathfrak{J}_\tau^{\text{test}}$ on M_τ . We choose $\mathfrak{J}_\tau^{\text{test}}$ as the push-forward of $\mathfrak{J}^{\text{test}}$ under F_τ , with an additional scalar component formed of the directional derivative of f_τ (the reason for this choice will become clear in Lemma 4.3.2).

More precisely,

$$\mathfrak{J}_\tau^{\text{test}} := \left\{ \left((F_\tau)_*(a + D_u \log f_\tau), (F_\tau)_*u \right) \text{ with } \mathbf{u} = (a, u) \in \mathfrak{J}^{\text{test}} \right\}, \quad (4.59)$$

where the push-forward is defined by

$$\begin{aligned} (F_\tau)_*a &: M_\tau \rightarrow \mathbb{R}^+, & ((F_\tau)_*a)(F_\tau(x)) &= a(x) \\ (F_\tau)_*u &: M_\tau \rightarrow T\mathcal{F}, & ((F_\tau)_*u)(F_\tau(x)) &= DF_\tau|_x u(x) \end{aligned} \quad (4.60)$$

(equivalently, the last relation can be written as $((F_\tau)_*u)|_{F_\tau(x)}\eta = u|_x(\eta \circ F_\tau)$ for any test function η defined in a neighborhood of $F_\tau(x)$).

We point out that the push-forward of $\mathfrak{J}^{\text{diff}}$ is in general not the same as the differentiable jets corresponding to the measure $\tilde{\rho}_\tau$, as is illustrated in the following example.

Example 4.3.1. (*the causal variational principle on the sphere*) The causal variational principle on the sphere is obtained from the setting of causal fermion systems by taking a mathematical simplification of a special case. It was introduced in [Fin10a, Section 1] (cf. [Fin10a, Examples 1.5, 1.6 and 2.8]) and analyzed in more detail in [FS13, Section 5]. We choose $\mathcal{F} = S^2$ and let $\mathcal{D} \in C^\infty(\mathcal{F} \times \mathcal{F}, \mathbb{R})$ be the smooth function

$$\mathcal{D}(x, y) := 2\tau^2 (1 + \langle x, y \rangle) (2 - \tau^2 (1 - \langle x, y \rangle)),$$

where $\tau \geq 1$ is a parameter of the model and $\langle \cdot, \cdot \rangle$ is the scalar product on \mathbb{R}^3 . Obviously, the function \mathcal{D} depends only on the angle $\vartheta \in [0, \pi]$ between the points $x, y \in S^2$ (defined by $\cos \vartheta = \langle x, y \rangle$). We here choose $\tau = \sqrt{2}$, so that

$$\mathcal{D} = \mathcal{D}(\vartheta) = 8 (1 + \cos \vartheta) \cos \vartheta.$$

The function \mathcal{D} has a maximum at $\vartheta = 0$ and changes signs at $\vartheta_{\max} := \frac{\pi}{2}$; more precisely

$$\mathcal{D}|_{[0, \vartheta_{\max})} > 0, \quad \mathcal{D}(\vartheta_{\max}) = 0, \quad \mathcal{D}|_{(\vartheta_{\max}, \pi]} \leq 0.$$

We define the Lipschitz-continuous *Lagrangian* \mathcal{L} by

$$\mathcal{L} = \max(0, \mathcal{D}) \in C^{0,1}(\mathcal{F} \times \mathcal{F}, \mathbb{R}_0^+). \quad (4.61)$$

Hence $\mathcal{L}(\vartheta)$ is positive if and only if $0 \leq \vartheta < \vartheta_{\max}$. Furthermore, \mathcal{L} is not differentiable at $\vartheta = \vartheta_{\max}$ since the semi-derivatives $\partial_\vartheta^+ \mathcal{L}(\vartheta)$ and $\partial_\vartheta^- \mathcal{L}(\vartheta)$ do not agree at this point.

It is shown in [FS13] that, for our choice of τ , a minimizer of the causal variational principle (4.1) is given by a normalized counting measure supported on an octahedron. Thus, denoting the set of unit vectors in \mathbb{R}^3 by $\mathbb{B} := \{e_1, e_2, e_3\}$, the measure

$$\rho = \frac{1}{6} \sum_{x \in \pm \mathbb{B}} \delta_x$$

is a minimizer (where δ_x denotes the Dirac measure supported at $x \in S^2$). Note that for

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all distinct points $x, y \in \text{supp } \rho$, the angle ϑ is either $\frac{\pi}{2}$ or π , implying that $\mathcal{L}(x, y) = 0$. As a consequence,

$$\ell(x) = \mathcal{D}(0) \quad \text{for all } x \in M.$$

In order to determine $\mathfrak{J}^{\text{diff}}$ as defined in (4.54), given any $x \in M$ and a non-zero vector $u \in T_x \mathcal{F}$, we let $\gamma : (-\delta, \delta) \rightarrow \mathcal{F}$ be a smooth curve with $\gamma(0) = x$ and $\dot{\gamma}(0) = u$. Qualitatively speaking, the function $\ell(\gamma(\tau))$ has a “cusp-like minimum” at $\tau = 0$ because for $\tau > 0$, there is at least one point $y \in M$ which contributes to $\ell(\gamma(\tau))$ whereas for $\tau < 0$, the same is true for a different point \tilde{y} . This can be made precise as follows. There is at least one point $y \in M$ with $\vartheta_{x,y} = \vartheta_{\max}$ and $\partial_\tau \vartheta_{\gamma(\tau),y} < 0$ at $\tau = 0$. This point contributes to ℓ for positive τ , i.e.

$$\ell(\gamma(\tau)) \geq \mathcal{D}(\vartheta_{\gamma(\tau),x}) + \mathcal{D}(\vartheta_{\gamma(\tau),y}) \quad \text{if } \tau \geq 0$$

and thus

$$\begin{aligned} D_u^+ \ell(x) &= \partial_\tau^+ \ell(\gamma(\tau)) \big|_{\tau=0} \geq \partial_\tau (\mathcal{D}(\vartheta_{\gamma(\tau),x}) + \mathcal{D}(\vartheta_{\gamma(\tau),y})) \big|_{\tau=0} \\ &= \mathcal{D}'(0) \partial_\tau \vartheta_{\gamma(\tau),x} \big|_{\tau=0} + \mathcal{D}'(\vartheta_{\max}) \partial_\tau \vartheta_{\gamma(\tau),y} > 0. \end{aligned}$$

Here, in the second step we used that \mathcal{D} is differentiable, and hence the one-sided derivatives agree with the derivative, and in the last step we used that $\mathcal{D}'(0) = 0$ and $\mathcal{D}'(\vartheta_{\max}) < 0$. Likewise, there is a point $\tilde{y} \in M$ with $\vartheta_{x,\tilde{y}} = \vartheta_{\max}$ and $\partial_\tau \vartheta_{\gamma(\tau),\tilde{y}} > 0$. This point contributes to ℓ for negative τ , implying that

$$\begin{aligned} D_{-u} \ell(x) &= -\partial_\tau^- \ell(\gamma(\tau)) \big|_{\tau=0} \geq -\partial_\tau (\mathcal{D}(\vartheta_{\gamma(\tau),x}) - \mathcal{D}(\vartheta_{\gamma(\tau),\tilde{y}})) \big|_{\tau=0} \\ &= -\mathcal{D}'(0) \partial_\tau \vartheta_{\gamma(\tau),x} \big|_{\tau=0} - \mathcal{D}'(\vartheta_{\max}) \partial_\tau \vartheta_{\gamma(\tau),\tilde{y}} > 0. \end{aligned}$$

Hence $D_u^+ \ell(x) \neq -D_{-u}^+ \ell(x)$, so that the directional derivative $D_u \ell(x)$ does not exist. We conclude that $\Gamma^{\text{diff}} = \{0\}$ and thus

$$\mathfrak{J}^{\text{diff}} = C^\infty(M) \oplus \{0\}. \quad (4.62)$$

We next define a family of measures $\tilde{\rho}_\tau$ as in (4.58). To this end, we choose trivial weight functions $f_\tau \equiv 1$, and choose a family of diffeomorphism which change the angles between the points of M in the sense that

$$\vartheta_{F_\tau(x), F_\tau(y)} \neq \vartheta_{\max} \quad \text{for all } x, y \in M \text{ and all } \tau \neq 0.$$

Then for any $\tau \neq 0$, the Lagrangian is smooth on a neighborhood of $M_\tau \times M_\tau$, so that

$$\mathfrak{J}^{\text{diff}}(\tilde{\rho}_\tau) = C^\infty(M_\tau) \oplus C^\infty(M_\tau, T\mathcal{F}).$$

On the other hand, using the formula in (4.59) to define the push-forward of $\mathfrak{J}^{\text{diff}}$ as given by (4.62), we obtain the jet space $C^\infty(M_\tau) \oplus \{0\}$. Hence the push-forward of $\mathfrak{J}^{\text{diff}}$ does

not coincide with the differentiable jets of the measure $\tilde{\rho}_\tau$. \diamond

In the next lemma we bring the weak EL equations for families of solutions into a convenient form. To this end, as in (4.36), we define

$$\ell_\tau(z) = \int_{\mathcal{F}} \mathcal{L}(z, y) d\rho_\tau(y) - \frac{\nu}{2}. \quad (4.63)$$

Lemma 4.3.2. *Assume that $(\tilde{\rho}_\tau)_{\tau \in (-\delta, \delta)}$ is a family of solutions of the weak EL equations (4.57) in the sense that*

$$\nabla_{\mathbf{u}(\tau)} \ell_\tau(z) = 0 \quad \text{for all } z \in M_\tau \text{ and } \mathbf{u}(\tau) \in \mathfrak{J}_\tau^{\text{test}}. \quad (4.64)$$

Then for any $\mathbf{u} \in \mathfrak{J}^{\text{test}}$ and all $\tau \in (-\delta, \delta)$,

$$0 = \nabla_{\mathbf{u}} \left(\int_M f_\tau(x) \mathcal{L}(F_\tau(x), F_\tau(y)) f_\tau(y) d\rho(y) - \frac{\nu}{2} f_\tau(x) \right). \quad (4.65)$$

Proof. Writing the jet $\mathbf{u}(\tau)$ as in (4.59) as

$$\mathbf{u}(\tau) = \left((F_\tau)_*(a + D_u \log f_\tau), (F_\tau)_* u \right) \quad (4.66)$$

with $\mathbf{u} \in \mathfrak{J}^{\text{test}}$ and using the definition of the push-forward (4.60), the weak EL equations (4.64) yield

$$0 = \nabla_{\tilde{\mathbf{u}}} \left(\int_M \mathcal{L}(F_\tau(x), F_\tau(y)) f_\tau(y) d\rho(y) - \frac{\nu}{2} \right) \quad (4.67)$$

with $\tilde{\mathbf{u}} = (a + (D_u \log f_\tau), u)$, valid for all $\tau \in (-\delta, \delta)$ and $x \in M$. Multiplying by $f_\tau(x)$, we obtain

$$\begin{aligned} 0 &= f_\tau(x) \nabla_{\tilde{\mathbf{u}}} \left(\int_M \mathcal{L}(F_\tau(x), F_\tau(y)) f_\tau(y) d\rho(y) - \frac{\nu}{2} \right) \\ &= f_\tau(x) \left(a(x) + (D_u \log f_\tau)(x) + D_u \right) \left(\int_M \mathcal{L}(F_\tau(x), F_\tau(y)) f_\tau(y) d\rho(y) - \frac{\nu}{2} \right) \\ &= (a(x) + D_u) f_\tau(x) \left(\int_M \mathcal{L}(F_\tau(x), F_\tau(y)) f_\tau(y) d\rho(y) - \frac{\nu}{2} \right) \\ &= \nabla_{\mathbf{u}} \left(\int_M f_\tau(x) \mathcal{L}(F_\tau(x), F_\tau(y)) f_\tau(y) d\rho(y) - \frac{\nu}{2} f_\tau(x) \right), \end{aligned}$$

making it possible to write (4.67) “more symmetrically” in the form (4.65). \square

Differentiating (4.65) naively with respect to τ , we obtain the *linearized field equations*

$$\langle \mathbf{u}, \Delta \mathbf{v} \rangle|_M = 0 \quad \text{for all } \mathbf{u} \in \mathfrak{J}^{\text{test}} \quad (4.68)$$

with

$$\langle \mathbf{u}, \Delta \mathbf{v} \rangle(x) := \nabla_{\mathbf{u}} \left(\int_M (\nabla_{1,\mathbf{v}} + \nabla_{2,\mathbf{v}}) \mathcal{L}(x, y) d\rho(y) - \nabla_{\mathbf{v}} \frac{\nu}{2} \right), \quad (4.69)$$

where \mathbf{v} is the jet $\mathbf{v} = (\dot{f}_0, \dot{F}_0)$. In order to see the connection to the linearized field equations in the smooth setting (4.32), we note that a formal computation using (4.45) gives

$$\langle \mathbf{u}, \Delta \mathbf{v} \rangle(x) = \nabla_{\mathbf{u}} \nabla_{\mathbf{v}} \ell(x) + \int_M \nabla_{1,\mathbf{u}} \nabla_{2,\mathbf{v}} \mathcal{L}(x, y) d\rho(y).$$

In the present lower semi-continuous setting, it is not clear if the derivatives exist, nor if the derivatives may be interchanged with the integrals. It turns out to be preferable to work with (4.69). In order to make sense of this expression, we need to impose conditions on \mathbf{v} . This leads us to the following definition:

Definition 4.3.3. *A jet $\mathbf{v} \in \mathfrak{J}$ is referred to as a **solution of the linearized field equations** (or for brevity a **linearized solution**) if it has the following properties:*

- (11) *For all $y \in M$ and all x in an open neighborhood of M , the following combination of derivatives exists,*

$$(\nabla_{1,\mathbf{v}} + \nabla_{2,\mathbf{v}}) \mathcal{L}(x, y) \in \mathbb{R}. \quad (4.70)$$

Here the combination of directional derivatives is defined by

$$(D_{1,v} + D_{2,v}) \mathcal{L}(x, y) := \frac{d}{d\tau} \mathcal{L}(F_\tau(x), F_\tau(y)) \Big|_{\tau=0},$$

where F_τ is the flow of the vector field v .

- (12) *Integrating the expression (4.70) over y with respect to the measure ρ , the resulting function (defined on an open neighborhood of M) is differentiable in the direction of every jet $\mathbf{u} \in \mathfrak{J}^{\text{test}}$ and satisfies the linearized field equations (4.68).*

The vector space of all linearized solutions is denoted by $\mathfrak{J}^{\text{lin}} \subset \mathfrak{J}$.

In order to illustrate the significance of the condition (4.70), we now give an example where the derivative in (4.70) does not exist.

Example 4.3.4. *(the causal variational principle on the sphere continued)* We return to the causal variational principle on the sphere as considered in Example 4.3.1. We first want to give an example where for \mathcal{L} as given in (4.61), the condition (4.70) is violated. To this end, we choose two points $x, y \in S^2$ such that $\vartheta_{x,y} = \vartheta_{\max}$. Furthermore, we choose a vector field v which vanishes in a neighborhood of y . Then

$$\mathcal{L}(F_\tau(x), F_\tau(y)) = \mathcal{L}(F_\tau(x), y) \quad (4.71)$$

(where F_τ is again the flow of the vector field v). Next, we choose $v(x)$ to be nonzero, tangential to the great circle joining x and y and pointing in the direction of smaller

geodesic distance to y . Then

$$D_{1,-v}^+ \mathcal{L}(x, y) = 0 \quad \text{but} \quad D_{1,v}^+ \mathcal{L}(x, y) = -\mathcal{D}'(\vartheta_{\max}) > 0.$$

Hence (4.71) is not differentiable at $\tau = 0$. We conclude that the derivative in (4.70) does not exist.

Despite the just-explained difficulty to satisfy the condition (4.70), the space $\mathfrak{J}^{\text{lin}}$ contains non-trivial vector fields. For example, if $(F_\tau)_{\tau \in (-\delta, \delta)}$ is a smooth family of isometries of the sphere (for example rotations around a fixed axis), then the corresponding jet $\mathfrak{v} = (0, v)$ with $v = \partial_\tau F_\tau|_{\tau=0}$ is in $\mathfrak{J}^{\text{lin}}$. This follows because

$$\frac{d}{d\tau} \mathcal{L}(F_\tau(x), F_\tau(y))|_{\tau=0} = 0,$$

i.e. the derivatives in (4.70) exist and are zero. It follows that condition (12), including the linearized field equations, are satisfied as well. This shows that $\mathfrak{v} \in \mathfrak{J}^{\text{lin}}$. We conclude that $\mathfrak{J}^{\text{lin}}$ is a vector space of dimension at least three.

In view of (4.62), this example also illustrates that $\mathfrak{J}^{\text{lin}}$ is in general not a subspace of $\mathfrak{J}^{\text{diff}}$. \diamond

4.3.3 The Symplectic Form and Hamiltonian Time Evolution

Following the construction in Section 4.2.3, we want to anti-symmetrize the linearized field equations (4.68) in the jets \mathfrak{u} and \mathfrak{v} . To this end, we now consider $\mathfrak{u}, \mathfrak{v} \in \mathfrak{J}^{\text{test}} \cap \mathfrak{J}^{\text{lin}}$. The conditions in Definition 4.3.3 ensure that the linearized field equations (4.68) are well-defined. For the construction of the symplectic form, we need additional technical assumptions. In order to make minimal assumptions, we work with semi-derivatives only. The existence of right semi-derivatives in $\mathbb{R} \cup \{\infty\}$ is guaranteed by condition (v) on page 96. We define the left semi-derivative as

$$D_{1,v}^- \mathcal{L}(x, y) := \lim_{\tau \nearrow 0} \frac{1}{\tau} \left(\mathcal{L}(\gamma(\tau), y) - \mathcal{L}(\gamma(0), y) \right) \in \mathbb{R} \cup \{\infty\}, \quad (4.72)$$

so that $D_{1,v}^- = -D_{1,-v}^+$. Similar to (4.52), we define

$$\nabla_{\mathfrak{u}}^- \ell(x) := a(x) \ell(x) + (D_{\mathfrak{u}}^- \ell)(x).$$

Before stating the additional assumptions, we explain why they are needed. First, we must ensure that the individual terms in (4.68) exist and that we may exchange the differentiation with integration. Second, when taking second derivatives, we must take into account that the jets are also differentiated. We use the notation

$$\nabla_{\mathfrak{u}(x)}^s \nabla_{1,\mathfrak{v}}^{s'} \mathcal{L}(x, y) \quad (4.73)$$

to indicate that the \mathfrak{u} -derivative also acts on \mathfrak{v} . With this notation, we can state the additional technical assumptions as follows:

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- (s1) The first and second semi-derivatives of the Lagrangian in the direction of $\mathfrak{J}^{\text{test}} \cap \mathfrak{J}^{\text{lin}}$ exist in \mathbb{R} . Moreover, for all x and y in a neighborhood of M , the symmetrized first semi-derivatives of the Lagrangian

$$(\nabla_{1,u}^+ + \nabla_{1,u}^-) \mathcal{L}(x, y)$$

are linear in $u \in \mathfrak{J}^{\text{test}} \cap \mathfrak{J}^{\text{lin}}$.

- (s2) The second semi-derivatives can be interchanged with the M -integration, i.e. for all $u, v \in \mathfrak{J}^{\text{test}}$ and $s, s' \in \{\pm\}$,

$$\begin{aligned} \int_M \nabla_{u(x)}^s \nabla_{1,v}^{s'} \mathcal{L}(x, y) d\rho(y) &= \nabla_u^s \int_M \nabla_{1,v}^{s'} \mathcal{L}(x, y) d\rho(y) \\ \int_M \nabla_{1,u}^s \nabla_{2,v}^{s'} \mathcal{L}(x, y) d\rho(y) &= \nabla_u^s \int_M \nabla_{2,v}^{s'} \mathcal{L}(x, y) d\rho(y). \end{aligned}$$

- (s3) For any $u, v \in \mathfrak{J}^{\text{test}} \cap \mathfrak{J}^{\text{lin}}$, the commutator $[u, v]$ is in $\mathfrak{J}^{\text{test}}$.

Theorem 4.3.5. *Under the above assumptions (11), (12) and (s1)–(s3), for any compact $\Omega \subset \mathcal{F}$, the surface layer integral*

$$\sigma_\Omega^{s,s'}(u, v) = \int_\Omega d\rho(x) \int_{M \setminus \Omega} d\rho(y) \sigma_{u,v}^{s,s'}(x, y) \quad (4.74)$$

with

$$\sigma_{u,v}^{s,s'}(x, y) := \nabla_{1,u}^s \nabla_{2,v}^{s'} \mathcal{L}(x, y) - \nabla_{1,v}^{s'} \nabla_{2,u}^s \mathcal{L}(x, y) \quad (4.75)$$

vanishes for all $u, v \in \mathfrak{J}^{\text{test}} \cap \mathfrak{J}^{\text{lin}}$ and all $s, s' \in \{\pm\}$.

Proof. Our starting point are the linearized field equations (4.68). Condition (s1) ensures that we can treat the terms of (4.69) independently if we take semi-derivatives. First, for $u, v \in \mathfrak{J}^{\text{lin}} \cap \mathfrak{J}^{\text{test}}$ we consider the term

$$\nabla_u^s \int_M \nabla_{1,v}^{s'} \mathcal{L}(x, y) d\rho(y) = \int_M \nabla_{u(x)}^s \nabla_{1,v}^{s'} \mathcal{L}(x, y) d\rho(y).$$

We now exchange u and v as well as s and s' and take the difference. Using the relation

$$\nabla_{1,v}^s = ss' \nabla_{1,ss'v}^{s'},$$

we obtain

$$\begin{aligned} \int_M (\nabla_{u(x)}^s \nabla_{1,v}^{s'} - \nabla_{v(x)}^{s'} \nabla_{1,u}^s) \mathcal{L}(x, y) d\rho(y) &= ss' \int_M \nabla_{1,ss'[u,v]}^s \mathcal{L}(x, y) d\rho(y) \\ &= ss' \nabla_{ss'[u,v]}^s \left(\ell(x) + \frac{\nu}{2} \right) = \nabla_{[u,v]} \left(\ell(x) + \frac{\nu}{2} \right) = \nabla_{[u,v]} \frac{\nu}{2} \end{aligned}$$

Here, in the second step we used the assumption (vi). In the third step, we applied assumptions (s3) and (4.56). In the last step, we used the weak EL equations.

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Using this result, anti-symmetrizing (4.69) (again by exchanging \mathbf{u} and \mathbf{v} as well as s and s') and using (4.68), we obtain the equations

$$\int_M \sigma_{\mathbf{u},\mathbf{v}}^{s,s'}(x,y) d\rho(y) = 0.$$

Integrating over Ω gives

$$\int_{\Omega} d\rho(x) \int_M d\rho(y) \sigma_{\mathbf{u},\mathbf{v}}^{s,s'}(x,y) = 0. \quad (4.76)$$

Moreover, it is obvious by the anti-symmetry of (4.75) that

$$\int_{\Omega} d\rho(x) \int_{\Omega} d\rho(y) \sigma_{\mathbf{u},\mathbf{v}}^{s,s'}(x,y) = 0.$$

Subtracting this equation from (4.76) gives the result. \square

Theorem 4.3.5 again makes it possible to introduce a Hamiltonian time evolution, just as explained in the introduction and in Section 4.2.3. The only difference compared to Section 4.2.3 is that, due to the semi-derivatives in (4.75), the surface layer integral (4.74) is in general not linear in the jets \mathbf{u} and \mathbf{v} . In order to obtain a bilinear form, we modify (4.42) as follows.

Proposition 4.3.6. *The mapping σ defined by*

$$\sigma : (\mathfrak{J}^{test} \cap \mathfrak{J}^{lin}) \times (\mathfrak{J}^{test} \cap \mathfrak{J}^{lin}) \rightarrow \mathbb{R}, \quad (\mathbf{u}, \mathbf{v}) \mapsto \frac{1}{4} \sum_{s,s'=\pm} \sigma_{\Omega_{N_t}}^{s,s'}(\mathbf{u}, \mathbf{v}) \quad (4.77)$$

is bilinear.

Proof. We have

$$\sigma(\mathbf{u}, \mathbf{v}) = \int_{\Omega_{N_t}} d\rho(x) \int_{M \setminus \Omega_{N_t}} d\rho(y) \sigma_{\mathbf{u},\mathbf{v}}(x,y)$$

with

$$\begin{aligned} \sigma_{\mathbf{u},\mathbf{v}}(x,y) &= \frac{1}{4} \sum_{s,s'=\pm} \sigma_{\mathbf{u},\mathbf{v}}^{s,s'}(x,y) \\ &= \frac{1}{4} \left(\nabla_{1,\mathbf{u}}^+ + \nabla_{1,\mathbf{u}}^- \right) \left(\nabla_{2,\mathbf{v}}^+ + \nabla_{2,\mathbf{v}}^- \right) \mathcal{L}(x,y) - \frac{1}{4} \left(\nabla_{1,\mathbf{v}}^+ + \nabla_{1,\mathbf{v}}^- \right) \left(\nabla_{2,\mathbf{u}}^+ + \nabla_{2,\mathbf{u}}^- \right) \mathcal{L}(x,y). \end{aligned}$$

The assumption (s1) and the symmetry of $\mathcal{L}(x,y)$ (condition (i) on page 82) imply that $(\nabla_{i,\mathbf{u}}^+ + \nabla_{i,\mathbf{u}}^-) \mathcal{L}(x,y)$ is linear in $\mathbf{u} \in \mathfrak{J}^{test} \cap \mathfrak{J}^{lin}$ for $i = 1, 2$. Hence $\sigma_{\mathbf{u},\mathbf{v}}(x,y)$ is indeed linear in \mathbf{u} and \mathbf{v} . \square

Since the other considerations at the end of Section 4.2.3 apply without changes, we do not repeat them here.

4.4 Example: A Lattice System in $\mathbb{R}^{1,1} \times S^1$

We now illustrate the previous constructions in a detailed example on two-dimensional Minkowski space $\mathbb{R}^{1,1}$. This example is inspired by physical field theories whose fields take values in S^1 , but it differs substantially from any such theory. Its main purpose is to allow for a test of the above constructions, it is not supposed to represent an actual model of a physical application of causal fermion systems. This is the case mainly because we choose a special Lagrangian \mathcal{L} instead of working with (4.23) or a special case thereof. Our choice of \mathcal{L} has the advantage that the minimizer is discrete, making the system suitable for a numerical analysis.

4.4.1 The Lagrangian

Let $(\mathbb{R}^{1,1}, \langle \cdot, \cdot \rangle)$ be two-dimensional Minkowski space. Thus, denoting the space-time points by $\underline{x} = (x^0, x^1)$ and \underline{y} , the inner product takes the form

$$\langle \underline{x}, \underline{y} \rangle = x^0 y^0 - x^1 y^1.$$

Moreover, let \mathcal{F} be the set

$$\mathcal{F} = \mathbb{R}^{1,1} \times S^1.$$

We denote points in $x \in \mathcal{F}$ by $x = (\underline{x}, x^\varphi)$ with $\underline{x} \in \mathbb{R}^{1,1}$ and $x^\varphi \in [-\pi, \pi)$. Next, we let A be the square

$$A = (-1, 1)^2 \subset \mathbb{R}^{1,1}.$$

Moreover, given $\varepsilon \in (0, \frac{1}{4})$, we let I be the the following subset of the interior of the light cones,

$$I = \{ \underline{x} \in \mathbb{R}^{1,1} \mid \langle \underline{x}, \underline{x} \rangle > 0 \text{ and } |x^0| < 1 + \varepsilon \}.$$

Furthermore, we let $f : \mathbb{R}^{1,1} \rightarrow \mathbb{R}$ be the function

$$f(\underline{x}) = \chi_{B_\varepsilon(0,1)}(\underline{x}) + \chi_{B_\varepsilon(0,-1)}(\underline{x}) - \chi_{B_\varepsilon(1,0)}(\underline{x}) - \chi_{B_\varepsilon(-1,0)}(\underline{x})$$

(where χ is the characteristic function and B_ε denotes the open Euclidean ball of radius ε in $\mathbb{R}^2 \simeq \mathbb{R}^{1,1}$). Finally, we let $V : S^1 \rightarrow \mathbb{R}$ be the function

$$V(\varphi) = 1 - \cos \varphi.$$

Given parameters $\delta > 0$, $\lambda_I \geq 2$ and $\lambda_A \geq 2\lambda_I + \varepsilon$, the Lagrangian \mathcal{L} is defined by

$$\begin{aligned} \mathcal{L}(x, y) = & \lambda_A \chi_A(\underline{x} - \underline{y}) + \lambda_I \chi_I(\underline{x} - \underline{y}) + V(x^\varphi - y^\varphi) f(\underline{x} - \underline{y}) \\ & + \delta \chi_{B_\varepsilon(0,0)}(\underline{x} - \underline{y}) V(x^\varphi - y^\varphi)^2. \end{aligned} \quad (4.78)$$

Lemma 4.4.1. *The function $\mathcal{L}(x, y)$ is non-negative and satisfies the conditions (i) and (ii) on page 82.*

Proof. The only negative contributions to $\mathcal{L}(x, y)$ arise in the term $V(x^\varphi - y^\varphi) f(\underline{x} - \underline{y})$

if $\underline{x} - \underline{y} \in B_\varepsilon(1, 0) \cup B_\varepsilon(-1, 0)$. For x and y with this property, we have

$$V(x^\varphi - y^\varphi) f(\underline{x} - \underline{y}) \geq -2 \quad \text{and} \quad \lambda_I \chi_I(\underline{x} - \underline{y}) \geq 2$$

because $V(S^1) \subset [0, 2] \subset \mathbb{R}$, $\lambda_I \geq 2$ and $B_\varepsilon(1, 0) \cup B_\varepsilon(-1, 0) \subset I$. We conclude that $\mathcal{L}(x, y) \geq 0$.

Condition (i) is satisfied because the sets A and I are point-symmetric around $\underline{x} = 0$, f is a sum of characteristic functions of sets which are mutually point-symmetric and $V(\varphi) = V(-\varphi)$. Condition (ii) is satisfied because V is continuous and because characteristic functions of open sets are lower semi-continuous. \square

4.4.2 A Local Minimizer

We next introduce a universal measure ρ supported on the unit lattice $\Gamma := \mathbb{Z}^2 \subset \mathbb{R}^{1,1}$ and show that for any $\delta > 0$, it is a local minimizer of the causal action in the sense of Definition 4.1.4.

Lemma 4.4.2. *The measure ρ given by*

$$\rho = \sum_{\underline{x} \in \Gamma} \delta_{(\underline{x}, 0)} \tag{4.79}$$

satisfies the conditions (iii) and (iv) (on page 82) as well as (v) and (vi) (on page 96).

Proof. Condition (iii) is satisfied because for every $x \in \mathcal{F}$, a neighborhood U with $\rho(U) < \infty$ is given for example by $U = B_\varepsilon(\underline{x}) \times (x^\varphi - \varepsilon, x^\varphi + \varepsilon) \subset \mathbb{R}^{1,1} \times S^1$. Condition (iv) is satisfied because the function $\mathcal{L}(x, \cdot)$ is bounded and has compact support (which implies ρ -integrability and boundedness of ℓ), and because ℓ is a finite sum of lower semi-continuous functions.

Condition (v) is satisfied because the generalized derivative of characteristic functions exists. Condition (vi) holds because $\ell(x)$ is a finite sum of terms of the form $\mathcal{L}(x, y)$. Hence differentiation and integration may be interchanged. \square

Clearly, the support of the above measure is given by

$$M := \text{supp } \rho = \Gamma \times \{0\} \subset \mathcal{F}. \tag{4.80}$$

Lemma 4.4.3. *The measure (4.79) satisfies the EL equations (4.10) if the parameter ν in (4.2) is chosen as*

$$\nu = 2\lambda_A + 4\lambda_I. \tag{4.81}$$

If $\delta > 0$, the implication (4.16) holds.

Proof. If $x \in M$, a direct computation using (4.78) and (4.2) shows that

$$\ell(x) = \lambda_A + 2\lambda_I - \frac{\nu}{2} = 0.$$

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Conversely, if $x \notin M$, then either $\underline{x} \notin \Gamma$ or $x^\varphi \neq 0$. In the first case, the characteristic function $\chi_A(\underline{x} - \cdot)$ equals one on at least two lattice points, implying that $\ell(x) \geq 2\lambda_A + 2\lambda_I - \frac{\nu}{2} = \lambda_A > 0$. In the remaining case $\underline{x} \in \Gamma$ and $x^\varphi \neq 0$, the term $\delta V(x^\varphi - 0)^2$ is non-negative, and it is strictly positive if $\delta > 0$. This concludes the proof. \square

This lemma shows that for $\delta > 0$, condition (a) in Proposition 4.1.7 is satisfied. The following Lemma shows that condition (c) holds as well:

Lemma 4.4.4. *Choosing $\lambda_A \geq 2\lambda_I + \varepsilon$, the inequality*

$$\langle \psi, \mathcal{L}_\rho \psi \rangle_\rho \geq \varepsilon \|\psi\|_\rho^2 \quad \text{holds for all } \psi \in \mathcal{D}(\mathcal{L}_\rho) .$$

Proof. For $\psi \in \mathcal{D}(\mathcal{L}_\rho)$ as defined by (4.14) and $e_t := (1, 0) \in \mathbb{R}^{1,1}$, we have

$$(\mathcal{L}_\rho \psi)(\underline{x}) = \lambda_A \psi(\underline{x}) + \lambda_I (\psi(\underline{x} + e_t) + \psi(\underline{x} - e_t))$$

and hence

$$\langle \psi, \mathcal{L}_\rho \psi \rangle_\rho = \lambda_A \|\psi\|_\rho^2 + \lambda_I \sum_{\underline{x} \in \Gamma} \overline{\psi(\underline{x})} (\psi(\underline{x} + e_t) + \psi(\underline{x} - e_t)) . \quad (4.82)$$

Applying Young's inequality

$$\left| \overline{\psi(\underline{x})} \psi(\underline{x} + e_t) \right| \leq \frac{1}{2} \left(|\psi(\underline{x} + e_t)|^2 + |\psi(\underline{x})|^2 \right) ,$$

the second term of (4.82) can be estimated by

$$\lambda_I \sum_{\underline{x} \in \Gamma} \overline{\psi(\underline{x})} (\psi(\underline{x} + e_t) + \psi(\underline{x} - e_t)) \geq -2\lambda_I \|\psi\|_\rho^2 .$$

Hence $\langle \psi, \mathcal{L}_\rho \psi \rangle_\rho \geq (\lambda_A - 2\lambda_I) \|\psi\|_\rho^2$, giving the result. \square

Corollary 4.4.5. *The measure ρ is a local minimizer of the causal action.*

Proof. Lemma 4.4.3 shows that condition (a) in Proposition 4.1.7 holds. Condition (b) follows because $\mathcal{L}(x, y)$ as defined by (4.78) is bounded on $\mathcal{F} \times \mathcal{F}$. Lemma 4.4.4 yields condition (c). \square

4.4.3 The Jet Spaces

We next determine the jet spaces. Clearly, in our setting of a discrete lattice (4.80), every function on M can be extended smoothly to a neighborhood of M . Thus the jet space (4.50) can be written as

$$\mathfrak{J}|_M = \{ \mathbf{u} = (a, u) \text{ with } a : M \rightarrow \mathbb{R} \text{ and } u : M \rightarrow T\mathcal{F} \} .$$

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When extending these jets to \mathcal{F} , for convenience we always choose an extension $\mathbf{u} : \mathcal{F} \rightarrow T\mathcal{F}$ which is locally constant on M . We denote the vector component by $u = (u^0, u^1, u^\varphi)$. In order to determine the differentiable jets, we recall from the the proof of Lemma 4.4.3 that

$$\begin{cases} \ell(\underline{x}, x^\varphi) = \delta V(x^\varphi)^2 & \text{if } \underline{x} \in \Gamma \\ \ell(\underline{x}, x^\varphi) \geq \lambda_A + \delta V(x^\varphi)^2 & \text{if } \underline{x} \notin \Gamma. \end{cases}$$

Hence the differentiable jets (4.54) are given by

$$\mathfrak{J}^{\text{diff}} = \{ \mathbf{u} = (a, u) \text{ with } a : M \rightarrow \mathbb{R} \text{ and } u = (0, 0, u^\varphi) : M \rightarrow T\mathcal{F} \}.$$

We choose $\mathfrak{J}^{\text{test}} = \mathfrak{J}^{\text{diff}}$.

Proposition 4.4.6. *The linearized solutions $\mathfrak{J}^{\text{lin}}$ of Definition 4.3.3 consist of all jets $\mathbf{v} = (b, v) \in \mathfrak{J}$ with the following properties:*

(A) *The scalar component $b : M \rightarrow \mathbb{R}$ satisfies the equation*

$$\lambda_A b(\underline{x}, 0) + \lambda_I (b(\underline{x} + e_t, 0) + b(\underline{x} - e_t, 0)) = 0. \quad (4.83)$$

(B) *The vector component $v : M \rightarrow T\mathcal{F}$ consists of a constant vector $\underline{v} \in \mathbb{R}^{1,1}$ and a function $v^\varphi : M \rightarrow \mathbb{R}$, i.e.*

$$v(x) = (\underline{v}, v^\varphi(x)),$$

where the function v^φ satisfies the discrete wave equation on Γ ,

$$\sum_{\underline{y} \in \Gamma} f(\underline{x} - \underline{y}) v^\varphi(\underline{y}, 0) = 0. \quad (4.84)$$

Proof. For ease in notation, we identify $M = \Gamma \times \{0\}$ with the lattice Γ . Our first aim is to show that that a jet $\mathbf{v} \in \mathfrak{J}$ satisfies condition (11) in Definition 4.3.3 if and only if it is of the form

$$\mathbf{v}(\underline{x}) = (b(\underline{x}), (\underline{v}, v^\varphi(\underline{x}))) \quad (4.85)$$

with a constant vector $\underline{v} \in \mathbb{R}^{1,1}$ and mappings $b, v^\varphi : \Gamma \rightarrow \mathbb{R}$.

Since (11) does not pose a condition on the scalar component, it suffices to consider the vector component $v = (\underline{v}, v^\varphi)$. Moreover, using that the Lagrangian is smooth in the variables x^φ and y^φ , it suffices to consider the component \underline{v} . If \underline{v} is a constant vector in Minkowski space, its flow does not change the difference vector $\underline{x} - \underline{y}$ in the Lagrangian (4.78). Therefore, the combination of directional derivatives in (4.70) exists and vanishes, implying that the jets of the form (4.85) satisfy the condition (11).

The following argument shows that for every jet satisfying (11) the component \underline{v} is indeed constant: Assume conversely that $\underline{v} : \Gamma \rightarrow \mathbb{R}^{1,1}$ is not constant. Then there are neighboring points $\underline{x}, \underline{y} \in \Gamma$ with $\underline{v}(\underline{x}) \neq \underline{v}(\underline{y})$. In order for the combination of directional derivatives in (4.70) to exist, the function $\mathcal{L}(F_\tau(x), F_\tau(y))$ must be differentiable in τ . This implies that the characteristic functions in (4.78) must be continuous at $\tau = 0$. We

first evaluate this condition if \underline{x} and \underline{y} are diagonal neighbors (i.e. $\underline{y} = \underline{x} + (\pm 1, \pm 1)$). In this case, for the characteristic function $\chi_I(F_\tau(x) - F_\tau(y))$ to be continuous, the vector $\underline{v}(\underline{x}) - \underline{v}(\underline{y})$ must be collinear to $\underline{x} - \underline{y}$. But then the continuity of the characteristic function $\chi_A((F_\tau(x) - F_\tau(y)))$ implies that $\underline{v}(\underline{x}) = \underline{v}(\underline{y})$. This a contradiction. We conclude that \underline{v} is constant on the even and odd sublattices of Γ . In the remaining case that \underline{v} describes a constant translation of the even sublattice relative to the odd sublattice, we can choose neighboring lattice points $\underline{x}, \underline{y} \in \Gamma$ on the odd and even sublattice, respectively, such that the vector $\underline{v}(\underline{x}) - \underline{v}(\underline{y})$ is non-zero and is not tangential to the discontinuity of the characteristic function χ_A . This implies that the function $\chi_A(F_\tau(x) - F_\tau(y))$ is not continuous at $\tau = 0$, which is again a contradiction.

We conclude that condition (11) in Definition 4.3.3 is satisfied precisely by all jets of the form (4.85). For such jets, the first part of condition (12) is satisfied because $\mathcal{L}(x, \cdot)$ has bounded support and is smooth in x^φ . It remains to evaluate the linearized field equations (4.68). For the constant component $\underline{v} \in \mathbb{R}^{1,1}$, (4.70) vanishes. Therefore we can set $\underline{v} = 0$ in the remainder of this proof. Since all our jets are locally constant, for $\mathbf{u} = (a, u) \in \mathfrak{J}^{\text{test}}$ and $x, y \in M$ we have

$$\begin{aligned} \nabla_{\mathbf{u}(x)} \nabla_{1,\mathbf{v}} \mathcal{L}(x, y) &= \left(a(\underline{x}) + u^\varphi(\underline{x}) \frac{\partial}{\partial x^\varphi} \right) \left(b(\underline{x}) + v^\varphi(\underline{x}) \frac{\partial}{\partial x^\varphi} \right) \mathcal{L}(x, y) \\ &\stackrel{(*)}{=} \left(a(\underline{x}) b(\underline{x}) + u^\varphi(\underline{x}) v^\varphi(\underline{x}) \frac{\partial^2}{\partial x^{\varphi 2}} \right) \mathcal{L}(x, y) \\ &= a(\underline{x}) b(\underline{x}) \mathcal{L}(x, y) - u^\varphi(\underline{x}) v^\varphi(\underline{x}) f(\underline{x} - \underline{y}), \end{aligned}$$

where in $(*)$ we used that the term involving the first derivative vanishes since $V(\varphi)$ is minimal at $\varphi = 0$. Similarly,

$$\nabla_{1,\mathbf{u}} \nabla_{2,\mathbf{v}} \mathcal{L}(x, y) = a(\underline{x}) b(\underline{y}) \mathcal{L}(x, y) + u^\varphi(\underline{x}) v^\varphi(\underline{y}) f(\underline{x} - \underline{y}). \quad (4.86)$$

Hence, for any $x \in M$, the linearized field equation (4.68) can be written as

$$\begin{aligned} (\lambda_A + 2\lambda_I) a(\underline{x}) b(\underline{x}) + \lambda_A a(\underline{x}) b(\underline{x}) + \lambda_I (a(\underline{x}) b(\underline{x} + e_t) + a(\underline{x}) b(\underline{x} - e_t)) \\ + u^\varphi(\underline{x}) \sum_{\underline{y} \in \Gamma} v^\varphi(\underline{y}) f(\underline{x} - \underline{y}) - a(\underline{x}) b(\underline{x}) \frac{\nu}{2}, \end{aligned} \quad (4.87)$$

where we used the fact that $\sum_{\underline{y} \in \Gamma} f(\underline{x} - \underline{y}) = 0$. Evaluating (4.87) for $\mathbf{u} = (a, 0) \in \mathfrak{J}^{\text{test}}$ and using (4.81) gives (4.83). Similarly, evaluating (4.87) for $\mathbf{u} = (0, u) \in \mathfrak{J}^{\text{test}}$ yields (4.84). \square

4.4.4 The Symplectic Form

We now construct the symplectic form. In preparation, we need to verify all technical assumptions.

Lemma 4.4.7. *The conditions (s1) to (s3) on page 103 are satisfied.*

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Proof. Condition (s1) follows because jets in $\mathfrak{J}^{\text{lin}} \cap \mathfrak{J}^{\text{test}}$ only have a v^φ -component and $\mathcal{L}(x, y)$ is smooth in x^φ and y^φ . Condition (s2) follows from definition (4.79) and the fact that the support of $\mathcal{L}(x, \cdot)$ is finite, so that the ρ -integration reduces to a finite sum. Condition (s3) is satisfied because for $\mathbf{u}, \mathbf{v} \in \mathfrak{J}^{\text{test}} \cap \mathfrak{J}^{\text{lin}}$, the commutator vanishes due to our choice of locally constant jets. \square

Next, in order to find an explicit expression for the symplectic form (4.74), we choose a constant time slice

$$N_t = \{\underline{x} \in \Gamma \mid x^0 = t\} \quad \text{with} \quad t \in \mathbb{Z}.$$

We let Ω_{N_t} be the past of N_t , i.e.

$$\Omega_{N_t} = \{\underline{x} \in \Gamma \mid x^0 \leq t\}.$$

Proposition 4.4.8. *If we choose $\mathfrak{J}^{\text{test}}$ as the jets with spacelike compact support,*

$$\mathfrak{J}^{\text{test}} = \{\mathbf{u} \in \mathfrak{J}^{\text{diff}} \mid \text{supp } \mathbf{u}|_{N_t} \text{ is a finite set for all } t \in \mathbb{Z}\},$$

the symplectic form (4.74) is given by

$$\begin{aligned} \sigma_{\Omega_{N_t}}(\mathbf{u}, \mathbf{v}) &= \lambda_I \sum_{\underline{x} \in N_t} (a(\underline{x}) b(\underline{x} + e_t) - a(\underline{x} + e_t) b(\underline{x})) \\ &\quad + \sum_{\underline{x} \in N_t} (u^\varphi(\underline{x} + e_t) v^\varphi(\underline{x}) - u^\varphi(\underline{x}) v^\varphi(\underline{x} + e_t)), \end{aligned} \tag{4.88}$$

where again $e_t := (1, 0) \in \mathbb{R}^{1,1}$.

Remark 4.4.9. Note that the second sum (4.88) is the usual symplectic form associated to a discrete version of the wave equation on $\mathbb{R}^{1,1}$. Namely, after adding the terms $v^\varphi(\underline{x}) u^\varphi(\underline{x}) - v^\varphi(\underline{x}) u^\varphi(\underline{x}) = 0$ to the each summand (and similarly for the scalar component) and taking a suitable limit $e_t \rightarrow 0$, we obtain

$$\sigma_{\Omega_{N_t}}(\mathbf{u}, \mathbf{v}) = \sum_{\underline{x} \in N_t} (\dot{u}^\varphi(\underline{x}) v^\varphi(\underline{x}) - u^\varphi(\underline{x}) \dot{v}^\varphi(\underline{x})) + \sum_{\underline{x} \in N_t} \lambda_I (a(\underline{x}) \dot{b}(\underline{x}) - \dot{a}(\underline{x}) b(\underline{x})),$$

where the dot denotes the discrete t -derivative. \diamond

Proof of Proposition 4.4.8. The proof of Proposition 4.4.6 still goes through if $\mathfrak{J}^{\text{test}}$ is restricted to jets with spatially compact support. Given $\mathbf{u}, \mathbf{v} \in \mathfrak{J}^{\text{test}} \cap \mathfrak{J}^{\text{lin}}$, by applying (4.86) and Proposition 4.4.6, we can compute (4.75) to obtain (for any choice of $s, s' \in \{\pm\}$)

$$\begin{aligned} \sigma_{\mathbf{u}, \mathbf{v}}(x, y) &= \nabla_{1, \mathbf{u}} \nabla_{2, \mathbf{v}} \mathcal{L}(x, y) - \nabla_{1, \mathbf{v}} \nabla_{2, \mathbf{u}} \mathcal{L}(x, y) \\ &= a(\underline{x}) b(\underline{y}) \mathcal{L}(x, y) - b(\underline{x}) a(\underline{y}) \mathcal{L}(x, y) + u^\varphi(\underline{x}) v^\varphi(\underline{y}) f(\underline{x} - \underline{y}) - v^\varphi(\underline{x}) u^\varphi(\underline{y}) f(\underline{x} - \underline{y}) \end{aligned}$$

(where we again identified $M = \Gamma \times \{0\}$ with Γ). Using that

$$\sigma_{\Omega_{N_t}}(\mathbf{u}, \mathbf{v}) = \sum_{\underline{x} \in \Omega_{N_t}} \sum_{\underline{y} \in M \setminus \Omega_{N_t}} \sigma_{\mathbf{u}, \mathbf{v}}((\underline{x}, 0), (\underline{y}, 0)) ,$$

we obtain (4.88). □

Appendix A: The Fréchet Manifold Structure of \mathcal{B}

As in Section 4.2 we assume that \mathcal{F} is a smooth manifold of dimension $m \geq 1$. We want to endow the set \mathcal{B} with the structure of a Fréchet manifold. The first step is to specify the topology on the set \mathcal{B} in (4.30). We choose the *compact-open topology* defined as follows. First, parametrizing the measures according to (4.29) by a pair $(f, F) \in C^\infty(\mathcal{F}, \mathbb{R}) \times C^\infty(\mathcal{F}, \mathcal{F})$, we can identify \mathcal{B} with a subset of the space of such pairs,

$$\mathcal{B} \subset C^\infty(\mathcal{F}, \mathbb{R}) \times C^\infty(\mathcal{F}, \mathcal{F}) .$$

Our task is to endow the sets $C^\infty(\mathcal{F}, \mathbb{R})$ and $C^\infty(\mathcal{F}, \mathcal{F})$ with the structure of a Fréchet manifold. Once this has been accomplished, the Fréchet structure on \mathcal{B} can be introduced simply by assuming that \mathcal{B} is a Fréchet submanifold of the product manifold $C^\infty(\mathcal{F}, \mathbb{R}) \times C^\infty(\mathcal{F}, \mathcal{F})$.

Being a vector space, the space $C^\infty(\mathcal{F}, \mathbb{R})$ can be endowed even with the structure of a Fréchet space. To this end, on \mathcal{F} we choose an at most countable atlas $(x_\lambda, U_\lambda)_{\lambda \in \Lambda}$ (with an index set $\Lambda \subset \mathbb{N}$) whose charts $x_\lambda : U_\lambda \rightarrow \mathbb{R}^m$ are defined on relative compact subsets $U_\lambda \subset \mathcal{F}$. We then consider the Fréchet topology induced by the C^k -norms in these charts, i.e.

$$\|f\|_{k, \lambda} := \|f \circ x_\lambda^{-1}\|_{C^k(x_\lambda(U_\lambda))} .$$

The resulting topology is metrizable. It is induced for example by the distance function

$$d(f, g) := \sum_{k=0}^{\infty} \sum_{\lambda \in \Lambda} 2^{-k-\lambda} \arctan \|f - g\|_{k, \lambda} .$$

In order to endow $C^\infty(\mathcal{F}, \mathcal{F})$ with the structure of a Fréchet manifold, we work locally in a neighborhood of a point $F \in C^\infty(\mathcal{F}, \mathcal{F})$. First, we refine the previous atlas (x_λ, U_λ) in such a way that the domains U_λ are all convex geodesic neighborhoods with respect to a chosen Riemannian metric g on \mathcal{F} . Moreover, we further refine this atlas to a new atlas $(y_\gamma, V_\gamma)_{\gamma \in \Gamma}$ in such a way that F maps the domains of the new charts to domains of the old charts, meaning that for every $\gamma \in \Gamma$ there is $\lambda(\gamma) \in \Lambda$ such that

$$F(V_\gamma) \subset U_{\lambda(\gamma)}$$

(for example, one can choose the domains of the new charts as open subsets of the sets $U_\lambda \cap F^{-1}(U_\nu)$ for $\lambda, \nu \in \Lambda$ and introduce the charts as the restrictions of x_λ to the

new domains). We restrict attention to mappings G which are so close to F that they map to the same charts, i.e.

$$G(V_\gamma) \subset U_{\lambda(\gamma)} \quad \text{for all } \gamma \in \Gamma. \quad (4.89)$$

For such mappings, we can define the C^k -norms by

$$\|G - F\|_{k,\gamma} = \|x_{\lambda(\gamma)} \circ G \circ y_\gamma^{-1} - x_{\lambda(\gamma)} \circ F \circ y_\gamma^{-1}\|_{C^k(y_\gamma(V_\gamma))}.$$

The resulting Fréchet topology is again metrizable, as becomes obvious for example by setting

$$d(F, G) = \begin{cases} 4 & \text{if (4.89) is violated} \\ \sum_{k=0}^{\infty} \sum_{\gamma \in \Gamma} 2^{-k-\lambda} \arctan \|F - G\|_{k,\gamma} & \text{if (4.89) holds.} \end{cases}$$

It remains to construct a local chart around F . To this end, it suffices to consider mappings G which satisfy (4.89). Then, since the domains of the charts U_λ are all geodesically convex, for any $x \in \mathcal{F}$ there is a unique vector $v(x) \in T_{F(x)}\mathcal{F}$ with the property that $G(x) = \exp_{F(x)} v(x)$. In this way, the mapping G can be described uniquely by a vector field $v \in C^\infty(F(\mathcal{F}), T\mathcal{F})$ on \mathcal{F} along $F(\mathcal{F})$. The mapping $G \rightarrow v$ is the desired chart, taking values in the linear space $C^\infty(F(\mathcal{F}), T\mathcal{F})$.

For clarity, we finally explain what the tangent vectors of \mathcal{B} are, how these tangent vectors act on functions, and how these derivatives are related to the derivative ∇_u as defined in (4.27). These elementary facts are also needed for the computation of the exterior derivative $d\gamma$ in the proof of Lemma 4.2.4. Given $\mathbf{v} = (b, v) \in T_\rho \mathcal{B}$, we let $\tilde{\rho}_\tau$ be a smooth curve in \mathcal{B} with $\tilde{\rho}_\tau|_{\tau=0} = \rho$ and $\dot{\rho}_\tau|_{\tau=0} = \mathbf{v}$. We again write the measures $\tilde{\rho}_\tau$ in the form (4.33) (see Lemma 4.2.2), so that (4.34) holds. Then the directional derivative of a smooth function ϕ on \mathcal{B} is defined as usual by

$$\mathbf{v}\phi = \frac{d}{d\tau} \phi(\tilde{\rho}_\tau)|_{\tau=0}.$$

In particular, the derivative of $\gamma(\mathbf{u})$ as defined in (4.44) is given by

$$\begin{aligned} \mathbf{v}\gamma(\mathbf{u})|_{\tilde{\rho}} &= \frac{d}{d\tau} \int_{\Omega_{N_t}} d\rho \int_{M \setminus \Omega_{N_t}} d\rho f_\tau(x) \nabla_{2,\mathbf{u}} \mathcal{L}(F_\tau(x), F_\tau(y)) f_\tau(y) \Big|_{\tau=0} \\ &= \int_{\Omega_{N_t}} d\rho \int_{M \setminus \Omega_{N_t}} d\rho (\nabla_{\mathbf{v}(x)} + \nabla_{\mathbf{v}(y)}) \nabla_{2,\mathbf{u}} \mathcal{L}(x, y). \end{aligned}$$

We point out that here the derivative $\nabla_{\mathbf{v}(y)}$ also acts on the jet \mathbf{u} in the derivative $\nabla_{2,\mathbf{u}}$. The commutator of such products of derivatives can be computed with the help of the following lemma.

Lemma 4.4.10. *For $\mathbf{u}, \mathbf{v} \in T_\rho \mathcal{B}$, we have*

$$\nabla_{[\mathbf{u}, \mathbf{v}]} = [\nabla_{\mathbf{u}}, \nabla_{\mathbf{v}}] . \quad (4.90)$$

Proof. Again denoting $\mathbf{u} = (a, u)$ and $\mathbf{v} = (b, v)$, for any smooth function η on \mathcal{F} we have

$$\begin{aligned} \nabla_{\mathbf{u}} \nabla_{\mathbf{v}} \eta(x) &= (a(x) + D_u)(b(x) + D_v) \eta(x) \quad \text{and} \\ [\nabla_{\mathbf{u}}, \nabla_{\mathbf{v}}] \eta(x) &= D_{[u, v]} \eta(x) + (D_u b)(x) \eta(x) - (D_v a)(x) \eta(x) . \end{aligned} \quad (4.91)$$

In order to compute the commutator $[\mathbf{u}, \mathbf{v}]$, we consider diffeomorphisms $\Phi_\tau, \tilde{\Phi}_s : \mathcal{B} \rightarrow \mathcal{B}$ along the vector fields \mathbf{u} and \mathbf{v} , i.e.

$$\Phi_0 = \text{id}, \quad \partial_\tau \Phi_\tau = \mathbf{u} \circ \Phi_\tau \quad \text{and} \quad \tilde{\Phi}_0 = \text{id}, \quad \partial_s \tilde{\Phi}_s = \mathbf{v} \circ \tilde{\Phi}_s .$$

Then

$$\begin{aligned} \int_{\mathcal{F}} \eta(x) d(\Phi_\tau \tilde{\Phi}_s \rho)(x) &= \int_{\mathcal{F}} f_\tau(x) \eta(F_\tau(x)) d(\tilde{\Phi}_s \rho)(x) \\ &= \int_{\mathcal{F}} \tilde{f}_s(x) f_\tau(\tilde{F}_s(x)) \eta(F_\tau(\tilde{F}_s(x))) d\rho(x) \end{aligned}$$

Differentiating with respect to s and τ at $\tau = s = 0$ gives

$$\begin{aligned} \int_{\mathcal{F}} \eta(x) d(\mathbf{u} \mathbf{v} \rho)(x) &= \frac{d^2}{d\tau ds} \int_{\mathcal{F}} \eta(x) d(\Phi_\tau \tilde{\Phi}_s \rho)(x) \Big|_{s=\tau=0} \\ &= \frac{d^2}{d\tau ds} \int_{\mathcal{F}} \tilde{f}_s(x) f_\tau(\tilde{F}_s(x)) \eta(F_\tau(\tilde{F}_s(x))) d\rho(x) \Big|_{s=\tau=0} \\ &= \int_{\mathcal{F}} \left(a(x) b(x) + (D_v a)(x) \right) \eta(x) d\rho(x) \\ &\quad + \int_{\mathcal{F}} \left(b(x) (D_u \eta)(x) + a(x) (D_v \eta)(x) \right) d\rho(x) \\ &\quad + \int_{\mathcal{F}} D_v D_u \eta(x) d\rho(x) . \end{aligned}$$

Likewise, exchanging the two diffeomorphism gives the vector $\mathbf{v} \mathbf{u} \rho$. Hence

$$\int_{\mathcal{F}} \eta(x) d([\mathbf{u}, \mathbf{v}] \rho)(x) = \int_{\mathcal{F}} \left(D_{[v, u]} \eta + (D_v a) \eta - (D_u b) \eta \right) d\rho(x) , \quad (4.92)$$

This shows that

$$[\mathbf{u}, \mathbf{v}] = (D_u b - D_v a, [u, v]) .$$

Comparing with (4.91) and the definition of $\nabla_{\mathbf{u}}$ in (4.27) gives (4.90).

We finally note for clarity that the minus sign in (4.92) arises because jets \mathbf{u}, \mathbf{v} act on functions on \mathcal{B} , whereas the derivatives $\nabla_{\mathbf{u}}$ and $\nabla_{\mathbf{v}}$ act on functions on M . When

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rewriting compositions of jets $\mathbf{u}\mathbf{v}$ as compositions of derivatives on M , the order of the composition is interchanged to $\nabla_v \nabla_u$. \square

Chapter 5

Stochastic and Non-Linear Correction Terms

In Chapter 4, we have established a formalism which allows to relate causal variational principles, the core analytic principle in the theory of causal fermion systems, to fields on space-time. These fields are jets $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathfrak{J}$ as defined in (4.49) and (4.50). We have found that the jets satisfy the linearized field equations (4.68), which in turn generate an evolution which is invariant w.r.t to an adaptation of a symplectic form to this setting (cf. Theorem 4.3.5). We referred to this evolution as *Hamiltonian time evolution*. The linearized field equations are conjectured to give rise to the fundamental equations of contemporary physics in the continuum limit.

The crucial step in the derivation of the linearized field equations is to test the Euler-Lagrange equations (4.10) weakly in directions of *differentiable jets* $\mathfrak{J}^{\text{diff}}$. Here, “differentiable” means that the directional derivative of the the function $\ell(x)$, as defined in (4.52), exists (cf. (4.53) and (4.54)). However, since the Lagrangian of the theory of causal fermion systems, given by (3.31), is only Lipschitz continuous, there may in general only be comparably few differentiable directions. This motivates the question of what happens to the linearized field equations if one tests in non-differentiable directions and which information is contained in the non-differentiable directions.

In this chapter, we show that extending the space of test-jets to include non-differentiable directions leads to the appearance of a stochastic term. Furthermore, we show that terms which are non-linear in the fields arise. As explained in Section 1.4, this supports the conjecture that in the continuum limit, the theory of causal fermion systems gives rise to a dynamical collapse theory. Furthermore, the correction terms might ultimately relate to experimental examination.

This chapter is organized as follows. In Section 5.1, we give preliminaries for this chapter, fix notation and introduce two technical assumptions. Section 5.2 is devoted to the derivation of the non-differentiable linearized field equations. A new term arises which has a stochastic nature. In Section 5.3, we show that a natural assumption can be made which implies that this term vanishes macroscopically. Section 5.4 contains the connection to the differentiable case of Chapter 4. In Section 5.5 we study the implications for the Hamiltonian time evolution. In Section 5.6 we prove the full non-differentiable field

equations to second order, which include contributions which are non-linear in the fields (Theorem 5.6.4). We conclude in Section 5.7 by extending a Noether-like theorem of Chapter 3 to the setting of this chapter.

5.1 Preliminaries

As in Chapter 4, for a given choice of *minimizing* measure ρ , we consider families of measures $(\rho_\tau)_{\tau \in (-\delta, \delta)}$ of the form

$$\rho_\tau = (F_\tau)_*(f_\tau \rho), \quad (5.1)$$

where f and F are smooth,

$$f \in C^\infty((-\delta, \delta) \times \mathcal{F} \rightarrow \mathbb{R}^+) \quad \text{and} \quad F \in C^\infty((-\delta, \delta) \times \mathcal{F} \rightarrow \mathcal{F}),$$

with $f_0(x) = 1$ and where F is a flow. (Details about flows are reviewed in Section 5.6.1. Here, it suffices to note that $F_0(x) = x$ and that $F_\tau := F(\tau, \cdot)$ is a diffeomorphism onto an open subset of \mathcal{F} .) As before, infinitesimally, the family (5.1) gives rise to a jet $\mathbf{v} = (\dot{f}_0, v) \in \mathfrak{J}$ where $\dot{f}_0 = \frac{d}{d\tau} f_\tau|_{\tau=0}$ and where v is the infinitesimal generator of F . We refer to \mathbf{v} as the *generator* of the family (5.1) and use dots to indicate also higher τ -derivatives (such as \ddot{f}_0). For the sake of brevity, we introduce the following abbreviations.

Notation 5.1.1. We define the notions of ‘integrability’, ‘almost everywhere’ and ‘measure zero’ to refer to the given measure ρ unless specified otherwise,

- ▶ ‘integrability’ := ρ -integrability,
- ▶ ‘almost everywhere’ := ρ -almost everywhere,
- ▶ ‘measure zero’ := measure zero with respect to ρ ,

and use the following shorthand notation for the right and left semi-derivatives,

$$\begin{aligned} \frac{d^+}{ds|_0} h(s) &= \frac{d^+}{ds} h(s)|_{s=0} = \lim_{s \searrow 0} \frac{h(s) - h(0)}{s} \\ \frac{d^-}{ds|_0} h(s) &= \frac{d^-}{ds} h(s)|_{s=0} = \lim_{s \nearrow 0} \frac{h(s) - h(0)}{s} = -\frac{d^+}{ds|_0} h(-s), \end{aligned} \quad (5.2)$$

where h is any function whose semi-derivatives exist. For any $x \in \mathcal{F}$ and $v \in T_x \mathcal{F}$, we define directional semi-derivatives as

$$D_v^\pm f(x) := \frac{d^\pm}{d\tau|_0} f(\gamma(\tau)), \quad (5.3)$$

where $\gamma \in C^1((-1, 1), \mathcal{F})$ is any curve with $\gamma(0) = x$ and $\gamma'(0) = v$, and we again indicate on which argument of a function $f(x, y)$ a semi-derivative acts by subscripts, $D_{i,v}^\pm f(x, y)$ for $i = 1, 2$. Furthermore, *semi-derivatives in direction of a jet* $\mathbf{v} = (b, v) \in \mathfrak{J}$ are defined

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as

$$\nabla_{\mathfrak{v}}^{\pm} := b(x) + D_v^{\pm} \quad (5.4)$$

and we denote *symmetric (directional) semi-derivatives* as

$$\frac{\tilde{d}}{ds}|_0 = \frac{1}{2} \left(\frac{d^+}{ds}|_0 + \frac{d^-}{ds}|_0 \right) \quad \text{and} \quad \tilde{D}_v := \frac{1}{2} (D_v^+ - D_{-v}^+) = \frac{1}{2} (D_v^+ + D_v^-), \quad (5.5)$$

as well as

$$\tilde{\nabla}_{\mathfrak{v}} := \frac{1}{2} (\nabla_{\mathfrak{v}}^+ - \nabla_{-\mathfrak{v}}^+) = \frac{1}{2} (\nabla_{\mathfrak{v}}^+ + \nabla_{\mathfrak{v}}^-). \quad (5.6)$$

This notation is compatible with the commutator of two jets $\mathfrak{v}, \mathfrak{w} \in \mathfrak{J}$, defined as

$$\nabla_{[\mathfrak{w}, \mathfrak{v}]}^+ := \nabla_{\mathfrak{w}}^+ \nabla_{\mathfrak{v}}^+ - \nabla_{\mathfrak{v}}^+ \nabla_{\mathfrak{w}}^+, \quad (5.7)$$

because

$$\begin{aligned} & \tilde{\nabla}_{\mathfrak{w}} \tilde{\nabla}_{\mathfrak{v}} - \tilde{\nabla}_{\mathfrak{v}} \tilde{\nabla}_{\mathfrak{w}} \\ &= \frac{1}{4} (\nabla_{\mathfrak{w}}^+ - \nabla_{-\mathfrak{w}}^+) (\nabla_{\mathfrak{v}}^+ - \nabla_{-\mathfrak{v}}^+) - \frac{1}{4} (\nabla_{\mathfrak{v}}^+ - \nabla_{-\mathfrak{v}}^+) (\nabla_{\mathfrak{w}}^+ - \nabla_{-\mathfrak{w}}^+) \\ &= \frac{1}{4} (\nabla_{[\mathfrak{w}, \mathfrak{v}]}^+ - \nabla_{[\mathfrak{w}, -\mathfrak{v}]}^+ - \nabla_{[-\mathfrak{w}, \mathfrak{v}]}^+ + \nabla_{[-\mathfrak{w}, -\mathfrak{v}]}^+) \\ &= \frac{1}{2} (\nabla_{[\mathfrak{w}, \mathfrak{v}]}^+ - \nabla_{-[\mathfrak{w}, \mathfrak{v}]}^+) = \tilde{\nabla}_{[\mathfrak{w}, \mathfrak{v}]}, \end{aligned}$$

and analogously for $\tilde{D}_{[w, v]}$.

In the case of repeated derivatives which act on the same argument, such as $\nabla_{i, \mathfrak{v}} \nabla_{i, \mathfrak{w}}$, we *always define the first derivative to act on the second derivative and on the corresponding argument of the function* (the i th argument in this case). E.g., with respect to the commutator (5.7) of two jets $\mathfrak{v} = (b, v)$, $\mathfrak{w} = (c, w) \in \mathfrak{J}$, this implies

$$\begin{aligned} \nabla_{[\mathfrak{w}, \mathfrak{v}]}^+ &= (c(x) + D_w^+) (b(x) + D_v^+) - (b(x) + D_v^+) (c(x) + D_w^+) \\ &= (D_w b) - (D_v c) + D_{[w, v]}^+, \end{aligned}$$

or written in terms of the notation (5.6),

$$\begin{aligned} \tilde{\nabla}_{[\mathfrak{w}, \mathfrak{v}]} &= (c(x) + \tilde{D}_w) (b(x) + \tilde{D}_v) - (b(x) + \tilde{D}_v) (c(x) + \tilde{D}_w) \\ &= (D_w b) - (D_v c) + \tilde{D}_{[w, v]}. \end{aligned} \quad (5.8)$$

We do not use the notation (4.73) in this chapter.

Finally, in the context of the family (5.1), for every $\Omega \in \Sigma(\mathcal{F})$, where $\Sigma(\mathcal{F})$ denotes the Borel sigma algebra of \mathcal{F} , we use the notation

$$\Omega_{\tau} := F_{\tau}(\Omega).$$

Throughout this chapter, we use \mathfrak{J} as defined in (4.49). I.e., we use jets which are

defined on \mathcal{F} rather than on M (where M is defined in (4.47)). \diamond

We work in the non-compact setting introduced in Section 4.1.1. In particular, we make (and refer to) Assumptions (i) and (ii) on page 82, as well as Assumptions (iii) and (iv). Assumptions (v) and (vi) on page 96 have to be strengthened in the present context to ensure well-definedness. One of the main differences to Chapter 4 is that in the latter, second semi-derivatives of $\mathcal{L}(x, y)$ are only required to exist in directions of differentiable jets (cf. Assumption (s1) on page 103), whereas here we demand that the second semi-derivatives exist for all jets.

To state the following assumptions, we define a function $h(x, y)$ to be

integrable in x and_c/or y

if it is integrable in x or y over every $\Omega \in \Sigma(\mathcal{F})$ and integrable in x and y for every $\Omega \times \Omega' \in \Sigma(\mathcal{F} \times \mathcal{F})$ whenever either Ω or Ω' is compact. (Note that integrability always refers to the chosen minimizer ρ , cf. Notation 5.1.1. The subscript ‘c’ stands for “compact”, indicating that one of two integrations needs to have compact domain.)

Assumption 5.1.2. For any $x, y \in \mathcal{F}$, $v \in \Gamma(T\mathcal{F})$, $F, G \in C^\infty((-\delta, \delta) \times \mathcal{F} \rightarrow \mathcal{F})$, $f, g \in C^\infty((-\delta, \delta) \times \mathcal{F} \rightarrow \mathbb{R}^+)$, $\tau \in (-\delta, \delta)$ and $h_\tau(x, y) := f_\tau(x) \mathcal{L}(F_\tau(x), G_\tau(y)) g_\tau(y)$ we assume that

(v) $h_\tau(x, y)$ is integrable in x and_c/or y and the semi-derivatives

$$\frac{d^+}{d\tau|_0} h_\tau(x, y) \quad \text{and} \quad D_{1,v}^+ \frac{d^+}{d\tau|_0} h_\tau(x, y) \quad (5.9)$$

as well as

$$\frac{d^{2+}}{d\tau^2|_0} h_\tau(x, y) \quad \text{and} \quad D_{1,v}^+ \frac{d^{2+}}{d\tau^2|_0} h_\tau(x, y) \quad (5.10)$$

exist and are integrable in x and_c/or y as well. Furthermore, we assume that $\frac{d^+}{d\tau|_{\tilde{\tau}}} \mathcal{L}(F_\tau(x), G_\tau(y)) = \frac{d^+}{d\tau|_{\tilde{\tau}}} \mathcal{L}(F_\tau(x), G_{\tilde{\tau}}(y)) + \frac{d^+}{d\tau|_{\tilde{\tau}}} \mathcal{L}(F_{\tilde{\tau}}(x), G_\tau(y))$ for all $\tilde{\tau} \in (-\delta, \delta)$.

(vi) the semi-derivatives in (5.9) and (5.10) can be exchanged with integration over x and_c/or y .

We give some direct consequences of the assumptions. First, note that (5.9) implies that for any $v \in T_x \mathcal{F}$, $D_{1,v}^+ \mathcal{L}(x, y)$ and $D_{2,v}^+ \mathcal{L}(x, y)$ exist in \mathbb{R} . Furthermore, choosing $f = 1 = g$, $G(x) = x$ and $F = \Phi$, where Φ is the flow of v , as well as $\Omega = M$, Assumption (vi) implies that

$$\int_M D_{1,v}^+ \mathcal{L}(x, y) d\rho(y) = D_{1,v}^+ \int_M \mathcal{L}(x, y) d\rho(y) = D_{1,v}^+ \ell(x)$$

for any $x \in \mathcal{F}$, where ℓ is defined in (4.45). Therefore, Assumptions (v) and (vi) of Chapter 4 (page 96) are satisfied. In the following, by (v) and (vi) we always refer to

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Assumptions 5.1.2. For completeness, we mention that Assumptions 5.1.2 also imply the first condition of Assumption (s1) as well as Assumption (s2) on page 103.

Finally, note that since ρ is a minimizer (cf. (5.1)), we have

$$D_v^+ \ell(x) \geq 0 \quad \text{but} \quad D_v^- \ell(x) \leq 0 \quad (5.11)$$

for any $x \in M$ and $v \in T_x \mathcal{F}$ (a consequence of the notation (5.2)), and equally so for $\nabla_u^+ \ell(x)$ and $\nabla_u^- \ell(x)$ with $u \in \mathfrak{J}$ (cf. (4.51)).

5.2 Non-Differentiable Linearized Field Equations

In this section, we derive field equations for the non-differentiable case. The next proposition gives the result. Note that ℓ_τ is defined in (4.63).

Proposition 5.2.1. *Let $\mathfrak{v} \in \mathfrak{J}$ be the generator of a family (5.1) of minimizers and $\mathfrak{w} \in \mathfrak{J}$ arbitrary. Then*

$$\frac{1}{2} (\nabla_{\mathfrak{w}}^+ - \nabla_{-\mathfrak{w}}^+) \int_M d\rho(y) (\nabla_{1,\mathfrak{v}}^+ + \nabla_{2,\mathfrak{v}}^+) \mathcal{L}(x, y) - \nabla_{\mathfrak{w}}^+ \nabla_{\mathfrak{v}}^+ \frac{\nu}{2} = \chi_{\mathfrak{w},\mathfrak{v}}(x) \quad (5.12)$$

for every $x \in M$, with

$$\chi_{\mathfrak{w},\mathfrak{v}}(x) = \frac{1}{2} \frac{d^+}{ds|_0} \frac{d^+}{d\tau|_0} f_\tau(x) \left(\ell_\tau(F_\tau(\Phi_s(x))) - \ell_\tau(F_\tau(\Phi_{-s}(x))) \right), \quad (5.13)$$

where Φ is the flow of the vectorial component w of \mathfrak{w} .

We refer to (5.12) as the **non-differentiable linearized field equations** and denote the set of all \mathfrak{v} which satisfy (5.12) for all $\mathfrak{w} \in \mathfrak{J}$ by $\mathfrak{J}^{\text{field}}$. (“Non-differentiable” indicates that the assumption of \mathfrak{w} being a differentiable jet is not necessary.) We give an interpretation of this result after its proof.

Proof. Choose $\mathfrak{v} = (b, v)$, $\mathfrak{w} = (c, w)$, f and F as in (5.1) and Φ as in the proposition. For every $x \in \mathcal{F}$, $\tau \in (-\delta, \delta)$ and $s \in (-s_{\max}, s_{\max})$ we have by definition of ℓ_τ

$$\begin{aligned} & \int_{M_\tau} d\rho_\tau(y) f_\tau(x) \left(\mathcal{L}(F_\tau(\Phi_s(x)), y) - \mathcal{L}(F_\tau(\Phi_{-s}(x)), y) \right) \\ &= f_\tau(x) \left(\ell_\tau(F_\tau(\Phi_s(x))) + \frac{\nu}{2} - \ell_\tau(F_\tau(\Phi_{-s}(x))) - \frac{\nu}{2} \right) \\ &= f_\tau(x) \left(\ell_\tau(F_\tau(\Phi_s(x))) - \ell_\tau(F_\tau(\Phi_{-s}(x))) \right), \end{aligned} \quad (5.14)$$

where the existence of the terms follows from Assumption (v). By definition of the directional semi-derivatives (5.3),

$$\frac{d^+}{d\tau|_0} \mathcal{L}(F_\tau(\Phi_s(x)), y) = D_{1,v|\Phi_s(x)}^+ \mathcal{L}(\Phi_s(x), y),$$

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where we have added the subscript $\Phi_s(x)$ to v merely to highlight that the derivative acts at the point $\Phi_s(x) \in \mathcal{F}$,

$$D_{1,v}^+ \mathcal{L}(\Phi_s(x), y) = D_{1,v|\Phi_s(x)}^+ \mathcal{L}(\Phi_s(x), y) .$$

Therefore,

$$\begin{aligned} & \frac{d^+}{ds|_0} \frac{d^+}{d\tau|_0} \int_M f_\tau(x) \mathcal{L}(F_\tau(\Phi_s(x)), F_\tau(y)) f_\tau(y) d\rho(y) \\ &= \frac{d^+}{ds|_0} \int_M \frac{d^+}{d\tau|_0} f_\tau(x) \mathcal{L}(F_\tau(\Phi_s(x)), F_\tau(y)) f_\tau(y) d\rho(y) \\ &= \frac{d^+}{ds|_0} \int_M (b(x) + D_{1,v|\Phi_s(x)}^+ + \nabla_{2,v}^+) \mathcal{L}(\Phi_s(x), y) d\rho(y) \\ &= D_w^+ \int_M (b(x) + D_{1,v}^+ + \nabla_{2,v}^+) \mathcal{L}(x, y) d\rho(y) - (D_w b)(x) \int_M \mathcal{L}(x, y) d\rho(y) \\ &= D_w^+ \int_M (\nabla_{1,v}^+ + \nabla_{2,v}^+) \mathcal{L}(x, y) d\rho(y) - (D_w b)(x) \int_M \mathcal{L}(x, y) d\rho(y) , \end{aligned} \tag{5.15}$$

where we have exchanged the differentiation with integration using Assumption (vi) and where the existence follows again from Assumption (v). For $x \in M$, this implies that

$$\begin{aligned} & \frac{d^+}{ds|_0} \frac{d^+}{d\tau|_0} \int_{M_\tau} d\rho_\tau(y) f_\tau(x) \left(\mathcal{L}(F_\tau(\Phi_s(x)), y) - \mathcal{L}(F_\tau(\Phi_{-s}(x)), y) \right) \\ &= \frac{d^+}{ds|_0} \frac{d^+}{d\tau|_0} \int_M d\rho(y) f_\tau(x) \left(\mathcal{L}(F_\tau(\Phi_s(x)), F_\tau(y)) - \mathcal{L}(F_\tau(\Phi_{-s}(x)), F_\tau(y)) \right) f_\tau(y) \\ &\stackrel{(5.15)}{=} (D_w^+ - D_{-w}^+) \int_M d\rho(y) (\nabla_{1,v}^+ + \nabla_{2,v}^+) \mathcal{L}(x, y) - 2 (D_w b)(x) \int_M d\rho(y) \mathcal{L}(x, y) \\ &= (D_w^+ - D_{-w}^+) \int_M d\rho(y) (\nabla_{1,v}^+ + \nabla_{2,v}^+) \mathcal{L}(x, y) - 2 (D_w b)(x) \frac{\nu}{2} , \end{aligned}$$

where we have used that $-D_{-w}^+ b = D_w^- b = D_w b$ since b is differentiable, and where in the last step we have used the definition of ℓ in (4.2) and $\ell|_M \equiv 0$ (which follows from ρ being a minimizer). Therefore, (5.14) gives

$$\begin{aligned} & \frac{1}{2} (D_w^+ - D_{-w}^+) \int_M d\rho(y) (\nabla_{1,v}^+ + \nabla_{2,v}^+) \mathcal{L}(x, y) - (D_w b)(x) \frac{\nu}{2} \\ &= \frac{1}{2} \frac{d^+}{ds|_0} \frac{d^+}{d\tau|_0} f_\tau(x) \left(\ell_\tau(F_\tau(\Phi_s(x))) - \ell_\tau(F_\tau(\Phi_{-s}(x))) \right) , \end{aligned} \tag{5.16}$$

where the existence of the right hand side follows from the existence of the left hand

side. Furthermore, we have

$$\begin{aligned}
 c(x) \int_M d\rho(y) (\nabla_{1,\mathfrak{v}}^+ + \nabla_{2,\mathfrak{v}}^+) \mathcal{L}(x, y) \\
 = c(x) \int_M d\rho(y) \frac{d^+}{d\tau|_0} f_\tau(x) \mathcal{L}(F_\tau(x), F_\tau(y)) f_\tau(y) \\
 = c(x) \frac{d^+}{d\tau|_0} f_\tau(x) \left(\ell_\tau(F_\tau(x)) + \frac{\nu}{2} \right),
 \end{aligned} \tag{5.17}$$

where existence follows again from Assumption (v). In the second step we have used Assumption (vi) and the definition of ℓ_τ in (4.63). Since \mathfrak{v} generates a family of minimizers, for every $x \in M$, we have

$$\ell_\tau(F_\tau(x)) \equiv 0 \quad \text{on } (-\delta, \delta),$$

which implies

$$f_\tau(x) \ell_\tau(F_\tau(x)) \equiv 0 \quad \text{on } (-\delta, \delta)$$

and therefore

$$\frac{d^+}{d\tau|_0} f_\tau(x) \ell_\tau(F_\tau(x)) = 0. \tag{5.18}$$

Hence (5.17) gives

$$c(x) \int_M d\rho(y) (\nabla_{1,\mathfrak{v}}^+ + \nabla_{2,\mathfrak{v}}^+) \mathcal{L}(x, y) - c(x) b(x) \frac{\nu}{2} = 0 \tag{5.19}$$

for every $x \in M$, where we have used that $b = \dot{f}_0$ due to our designation of \mathfrak{v} as (b, v) . Adding (5.19) and (5.16) gives the result. \square

To interpret Proposition 5.2.1, note that the left hand side of (5.12) is equal to the left hand side of the linearized field equations (4.68) if $\int_M d\rho(y) (\nabla_{1,\mathfrak{v}}^+ + \nabla_{2,\mathfrak{v}}^+) \mathcal{L}(x, y)$ exists and is differentiable. Therefore, (5.12) constitutes a generalization of the linearized field equations (4.68) to include testing with jets $\mathfrak{w} \in \mathfrak{J}$ which need not be elements of $\mathfrak{J}^{\text{diff}}$. We investigate the precise relation to the differentiable case in Section 5.4 below.

This brings us to the interpretation of the term $\chi_{\mathfrak{w},\mathfrak{v}}(x)$. Expanding it as

$$\begin{aligned}
 \chi_{\mathfrak{w},\mathfrak{v}}(x) &= \frac{1}{2} b(x) \frac{d^+}{ds|_0} \left(\ell(\Phi_s(x)) - \ell(\Phi_{-s}(x)) \right) \\
 &\quad + \frac{1}{2} \frac{d^+}{ds|_0} \frac{d^+}{d\tau|_0} \left(\ell_\tau(F_\tau(\Phi_s(x))) - \ell_\tau(F_\tau(\Phi_{-s}(x))) \right),
 \end{aligned} \tag{5.20}$$

we see that $\chi_{\mathfrak{w},\mathfrak{v}}(x)$ is related to the appearance of symmetric directional semi-derivatives $\tilde{D}_w \ell$ as defined in (5.5). Due to (5.11), for $x \in M$, $\tilde{D}_w \ell(x)$ in general has an arbitrary sign, depending on the exact numerical values of $D_w^+ \ell(x)$ and $D_w^- \ell(x)$ at the space-time point x . Since ℓ is in general only Lipschitz-continuous, $D_w^+ \ell(x)$ and $D_w^- \ell(x)$ need not

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even be continuous in x . Therefore, we are led to interpret $\chi_{\mathbf{w},\mathbf{v}}(x)$ as a *stochastic term*, fluctuating with varying sign as x changes. (The connection to random variables, which is implicit in this name, is intentional: If the microscopic structure of space-time, and hence the point-wise behaviour of ℓ , is not known, the term effectively appears as a random contribution and can be modelled as a random variable.) Due to this interpretation, in the following, we refer to $\chi_{\mathbf{w},\mathbf{v}}(x)$ as “the stochastic term”.

The mathematical side of this interpretation is supported by the next example. In the following sections, we study the behaviour of the stochastic term in more detail.

Example 5.2.2. Due to the EL equations (4.10), for fixed $x \in M$, the function $\ell(\Phi_s(x))$ has a cusp-like behaviour as s varies (cf. also (5.11)). In a first approximation, this behaviour is similar to the behaviour of a function $l : \mathbb{R} \rightarrow \mathbb{R}$ given by

$$l(x) = \begin{cases} \alpha x & x \geq 0 \\ -\beta x & x < 0, \end{cases}$$

where $\alpha, \beta \in \mathbb{R}$, $\alpha, \beta > 0$ and $x \in \mathbb{R}$. We use this function to illustrate the behaviour of the stochastic term (5.13). Denote by w a unit-vector field on \mathbb{R} so that we have $D_w^+ f = \partial^+ f$ for any $f : \mathbb{R} \rightarrow \mathbb{R}$, where ∂^+ denotes the right semi-derivative in (5.2). This gives

$$D_w^+ l(x) = \begin{cases} \alpha & x > 0 \\ \alpha & x = 0 \\ -\beta & x < 0, \end{cases} \quad D_{-w}^+ l(x) = \begin{cases} -\alpha & x > 0 \\ \beta & x = 0 \\ \beta & x < 0 \end{cases}$$

and

$$D_w^- l(x) = \begin{cases} \alpha & x > 0 \\ -\beta & x = 0 \\ -\beta & x < 0. \end{cases}$$

This shows that

$$(D_w^+ - D_{-w}^+)l(0) = \alpha - \beta,$$

which changes sign depending on the particular values of α and β , indicating that the first line of (5.20) has varying sign depending on the behaviour of the function $\ell(\Phi_s(x))$ at the space-time point $x \in M$.

We can include a variation (5.1) of minimizers in this model by promoting α and β to functions of a parameter τ ,

$$l_\tau(x) = \begin{cases} \alpha_\tau x & x \geq 0 \\ -\beta_\tau x & x < 0. \end{cases}$$

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Assuming that the semi-derivatives $\dot{\alpha}_0^+ := \frac{d^+}{d\tau|_0} \alpha_\tau$ and $\dot{\beta}_0^+ := \frac{d^+}{d\tau|_0} \beta_\tau$ exist, we have

$$\frac{d^+}{d\tau|_0} l_\tau(x) = \begin{cases} \dot{\alpha}_0^+ x & x \geq 0 \\ -\dot{\beta}_0^+ x & x < 0, \end{cases}$$

giving

$$(D_w^+ - D_{-w}^+) \frac{d^+}{d\tau|_0} l_\tau(0) = \dot{\alpha}_0^+ - \dot{\beta}_0^+.$$

In general, based on the theory of causal fermion systems, there is no reason to postulate a particular sign of $\dot{\alpha}_0^+$ or $\dot{\beta}_0^+$. Hence also the second line in (5.20) does not have a fixed sign. (We come back to this point at the end of Section 5.3.)

5.3 Stochastic Term Vanishes Macroscopically

In the last section, we have seen that a stochastic term appears in the non-differentiable linearized field equations. In this section, we show that a typical physical assumption implies that the stochastic term vanishes when averaged over macroscopic regions. Even so this assumption is not enforced by the first principles of the theory, it seems to be supported a posteriori by the fact that most contemporary experiments seem to be aligned with the predictions of contemporary physics within experimental errors.

Before giving the assumption, we define a minimal notion of “macroscopicity” in the context of causal fermion systems. The idea is that for a given physical situation which is described by the theory, the data required by the following definition is given.

Definition 5.3.1. (Macroscopicity) *We assume that the macroscopic regions of a minimizer ρ of the causal variational principle are described by a subset $\mathcal{M} \subset \Sigma(\mathcal{F})$. Furthermore we assume that for a family (5.1) of minimizers, the macroscopic regions are invariant with respect to the diffeomorphisms F_τ . (I.e., for every $\tilde{\Omega} \in \mathcal{M}$ and every $\tau \in (-\delta, \delta)$, we have $F_\tau(\tilde{\Omega}) \in \mathcal{M}_\tau$, where $\mathcal{M}_\tau \subset \Sigma(\mathcal{F})$ denotes the macroscopic region associated with ρ_τ .)*

The next definition captures the physical intuition that when averaged over a macroscopic region, fluctuations from the different space-time points might cancel each other, or “average out”. Here, this means that the sign-varying terms $\frac{1}{2}(D_w^+ \ell - D_{-w}^+ \ell)(x)$ at different space-time points x cancel each other’s contribution to an integration over a macroscopic region $\tilde{\Omega}$.

Definition 5.3.2. (Symmetric derivatives vanish macroscopically) *We assume that for every minimizer ρ of the causal variational principle, every $\Phi \in C^\infty(\mathcal{F} \times (-\delta, \delta) \rightarrow \mathcal{F})$ and every compact macroscopic region $\tilde{\Omega} \in \mathcal{M}$,*

$$\frac{1}{2} \frac{d^+}{ds|_0} \int_{\tilde{\Omega}} \left(\ell(\Phi_s(x)) - \ell(\Phi_{-s}(x)) \right) d\rho(x) = 0. \quad (5.21)$$

Note that the restriction to compact $\tilde{\Omega}$ arises because Assumption (v) guarantees existence of the left hand side of (5.21) only if $\tilde{\Omega}$ is compact.

Remark 5.3.3. (SYMMETRIC DERIVATIVES VANISH APPROXIMATELY) One might wonder why the symmetric derivatives should vanish exactly, as compared to vanishing approximately, when integrated over a macroscopic region. Indeed, this is not crucial, the right hand side of (5.21) could be replaced by ε for any $\varepsilon \in \mathbb{R}_0^+$, or by $\rho(\tilde{\Omega})^{-1}$, thus requiring that the symmetric derivatives only vanish approximately. The following proposition would still hold, with the right hand side of (5.22) replaced by ε or $\rho(\tilde{\Omega})^{-1}$, respectively. \diamond

Remark 5.3.4. (RELATION TO MACROSCOPIC DIFFERENTIABILITY) We note that even though (5.21) implies that the derivative $\frac{d}{ds}|_0 \int_{\tilde{\Omega}} (\ell(\Phi_s(x)) - \ell(\Phi_{-s}(x))) d\rho(x)$ exists and vanishes, this does not imply that the individual terms $\frac{d}{ds}|_0 \int_{\tilde{\Omega}} \ell(\Phi_s(x)) d\rho(x)$ exist. Therefore, Definition 5.3.2 is considerably weaker than the requirement of minimizers to be macroscopically differentiable in the sense that $\frac{d}{ds}|_0 \int_{\tilde{\Omega}} \ell(\Phi_s(x)) d\rho(x)$ exists. \diamond

The next proposition shows that the assumption of Definition 5.3.2 is sufficient to conclude that the stochastic term vanishes macroscopically.

Proposition 5.3.5. *Assume that symmetric derivatives vanish macroscopically and let $\mathfrak{v} \in \mathfrak{J}$ be the generator of a family of minimizers (5.1). Then for every $\mathfrak{w} \in \mathfrak{J}$ and every compact $\tilde{\Omega} \in \mathcal{M}$,*

$$\int_{\tilde{\Omega}} \chi_{\mathfrak{w}, \mathfrak{v}}(x) d\rho(x) = 0. \quad (5.22)$$

Proof. For every $u \in \Gamma(T\mathcal{F})$, Definition 5.3.2 and Assumption (vi) imply

$$\frac{1}{2} \int_{\tilde{\Omega}} (D_u^+ - D_{-u}^+) \ell(x) d\rho(x) = 0. \quad (5.23)$$

Denote $\mathfrak{w} = (c, w)$ and let Φ be the flow of w . Since $(\rho_\tau)_{\tau \in (-\delta, \delta)}$, as defined in (5.1), is a family of minimizers by assumption and since $\tilde{\Omega}$ compact implies that $\tilde{\Omega}_\tau$ is compact, (5.21), evaluated for ρ_τ , reads

$$\frac{1}{2} \frac{d^+}{ds}|_0 \int_{\tilde{\Omega}_\tau} (\ell_\tau(\Phi_s(x)) - \ell_\tau(\Phi_{-s}(x))) d\rho_\tau(x) = 0 \quad \text{for all } \tau \in (-\delta, \delta), \quad (5.24)$$

which in turn implies

$$\frac{1}{2} \frac{d^+}{d\tau}|_0 \frac{d^+}{ds}|_0 \int_{\tilde{\Omega}_\tau} (\ell_\tau(\Phi_s(x)) - \ell_\tau(\Phi_{-s}(x))) d\rho_\tau(x) = 0. \quad (5.25)$$

The existence of this integral is guaranteed by Assumption (v). We have

$$\begin{aligned}
 & \frac{d^+}{d\tau|_0} \frac{d^+}{ds|_0} \int_{\tilde{\Omega}} f_\tau(x) \ell_\tau(\Phi_s(F_\tau(x))) d\rho(x) \\
 &= \frac{d^+}{d\tau|_0} \int_{\tilde{\Omega}} f_\tau(x) D_{w|F_\tau(x)}^+ \ell_\tau(F_\tau(x)) d\rho(x) \\
 &= \int_{\tilde{\Omega}} (b(x) D_w^+ + D_v^+ D_w^+) \ell(x) d\rho(x) + \int_{\tilde{\Omega}} \frac{d^+}{d\tau|_0} D_w^+ \ell_\tau(x) d\rho(x) \\
 &= \int_{\tilde{\Omega}} (b(x) D_w^+ + D_w^+ D_v^+) \ell(x) d\rho(x) \\
 &\quad + \int_{\tilde{\Omega}} \frac{d^+}{d\tau|_0} D_w^+ \ell_\tau(x) d\rho(x) + \int_{\tilde{\Omega}} D_{[v,w]}^+ \ell(x) d\rho(x) \\
 &= \frac{d^+}{ds|_0} \int_{\tilde{\Omega}} (b(x) \ell(\Phi_s(x)) + D_{v|\Phi_s(x)}^+ \ell(\Phi_s(x))) d\rho(x) \\
 &\quad + \int_{\tilde{\Omega}} \frac{d^+}{d\tau|_0} \frac{d^+}{ds|_0} \ell_\tau(\Phi_s(x)) d\rho(x) + \int_{\tilde{\Omega}} D_{[v,w]}^+ \ell(x) d\rho(x) \\
 &= \frac{d^+}{ds|_0} \frac{d^+}{d\tau|_0} \int_{\tilde{\Omega}} f_\tau(x) \ell_\tau(F_\tau(\Phi_s(x))) d\rho(x) + \int_{\tilde{\Omega}} D_{[v,w]}^+ \ell(x) d\rho(x),
 \end{aligned} \tag{5.26}$$

where we have used Assumption (vi) to exchange differentiation and integration, and where in the last step we have used

$$\frac{d^+}{d\tau|_0} \frac{d^+}{ds|_0} \ell_\tau(\Phi_s(x)) = \frac{d^+}{ds|_0} \frac{d^+}{d\tau|_0} \ell_\tau(\Phi_s(x)), \tag{5.27}$$

which holds because the derivatives act on different variables (apparent upon expanding ℓ_τ). Therefore, (5.25) implies

$$\int_{\tilde{\Omega}} \chi_{\mathfrak{w},\mathfrak{v}}(x) d\rho(x) = \frac{1}{2} \int_{\tilde{\Omega}} (D_{[w,v]}^+ - D_{-[w,v]}^+) \ell(x) d\rho(x).$$

Using (5.23) for $u = [w, v]$ gives the result. \square

We conclude this section with the remark that Proposition 5.3.5 supports the interpretation of $\chi_{\mathfrak{w},\mathfrak{v}}$ as a stochastic term. To this end, recall that Example 5.2.2 shows that the first line in (5.20) has varying sign, but only implies that the second line in (5.20) does not have a fixed sign (since $\dot{\alpha}_0^+$ and $\dot{\beta}_0^+$ do not both have the same sign in general). The previous proposition shows that if the symmetric derivatives (5.21) vanish macroscopically, also the second line in (5.20) vanishes macroscopically. This implies that it also needs to have varying sign in general.

Furthermore, we remark that the assumption in Definition 5.3.2 is also supported by Proposition 5.7.2 in Section 5.7, where we show that if symmetric derivatives vanish macroscopically, the conservation laws of Chapter 3 also apply in the non-differentiable case considered here.

5.4 Connection to the Differentiable Case

In this section, we study the connection to the differentiable case established in Chapter 4. We give two different sets of assumptions which imply that the stochastic term (5.13) vanishes pointwise. In our first approach, we adapt the Definition (4.53) of differentiable jets to a family (5.1). In the second approach, we work with (4.59) to extend differentiable jets $\mathfrak{J}^{\text{diff}}$ to $\tau \in (-\delta, \delta)$ using the push-forward (4.60).

Definition 5.4.1. *A jet $\mathfrak{w} \in \mathfrak{J}$ is **differentiable with respect to a family** $(\rho_\tau)_{\tau \in (-\delta, \delta)}$ of measures if for all $\tau \in (-\delta, \delta)$ and all $x \in M$ we have*

$$(\nabla_{\mathfrak{w}}^+ \ell_\tau)(F_\tau(x)) = (-\nabla_{-\mathfrak{w}}^+ \ell_\tau)(F_\tau(x)) .$$

Proposition 5.4.2. *Let $\mathfrak{w} \in \mathfrak{J}$ be differentiable with respect to a family of minimizers of the form (5.1) whose generator is \mathfrak{v} . If $[\mathfrak{w}, \mathfrak{v}] \in \mathfrak{J}^{\text{diff}}$, for every $x \in M$, we have*

$$\chi_{\mathfrak{w}, \mathfrak{v}}(x) = 0 .$$

Proof. Denote $\mathfrak{w} = (c, w)$ and let Φ be the flow of w . Since \mathfrak{w} is differentiable with respect to the family (5.1), for every $x \in M$ and all $\tau \in (-\delta, \delta)$, $(D_w \ell_\tau)(F_\tau(x))$ exists. Since the family consist of minimizers, we have

$$(D_w \ell_\tau)(F_\tau(x)) = 0 \tag{5.28}$$

and therefore

$$\frac{1}{2} \frac{d^+}{ds|_0} \left(\ell_\tau(\Phi_s(F_\tau(x))) - \ell_\tau(\Phi_{-s}(F_\tau(x))) \right) = \frac{d}{ds|_0} \ell_\tau(\Phi_s(F_\tau(x))) = 0$$

for all $\tau \in (-\delta, \delta)$. This implies

$$\frac{1}{2} \frac{d^+}{d\tau|_0} \frac{d^+}{ds|_0} f_\tau(x) \left(\ell_\tau(\Phi_s(F_\tau(x))) - \ell_\tau(\Phi_{-s}(F_\tau(x))) \right) = 0 . \tag{5.29}$$

Arguing as in (5.26), we have

$$\begin{aligned}
 \frac{d^+}{d\tau|_0} \frac{d^+}{ds|_0} f_\tau(x) \ell_\tau(\Phi_s(F_\tau(x))) &= \frac{d^+}{d\tau|_0} f_\tau(x) D_{w|F_\tau(x)}^+ \ell_\tau(F_\tau(x)) \\
 &= (b(x) D_w^+ + D_v^+ D_w^+) \ell(x) + \frac{d^+}{d\tau|_0} D_w^+ \ell_\tau(x) \\
 &= (b(x) D_w^+ + D_w^+ D_v^+) \ell(x) + \frac{d^+}{d\tau|_0} D_w^+ \ell_\tau(x) + D_{[v,w]}^+ \ell(x) \\
 &= \frac{d^+}{ds|_0} (b(x) \ell(\Phi_s(x)) + D_{v|\Phi_s(x)}^+ \ell(\Phi_s(x)) + \frac{d^+}{d\tau|_0} \frac{d^+}{ds|_0} \ell_\tau(\Phi_s(x)) + D_{[v,w]}^+ \ell(x) \\
 &= \frac{d^+}{ds|_0} \frac{d^+}{d\tau|_0} f_\tau(x) \ell_\tau(F_\tau(\Phi_s(x))) + D_{[v,w]}^+ \ell(x) ,
 \end{aligned}$$

where in the last step we have again used (5.27). Therefore, (5.29) implies

$$\chi_{\mathfrak{w},\mathfrak{v}}(x) = \frac{1}{2} (D_{[w,v]}^+ \ell(x) - D_{-[w,v]}^+ \ell(x)) . \quad (5.30)$$

(We remark that this equation can also be obtained by noting that (5.28) implies

$$\begin{aligned}
 0 &= \frac{d}{d\tau|_0} \frac{d}{ds|_0} \ell_\tau(\Phi_s(F_\tau(x))) = \frac{d}{d\tau|_0} D_{w|F_\tau(x)} \ell_\tau(F_\tau(x)) \\
 &= D_v D_w \ell(x) + \frac{d}{d\tau|_0} D_w \ell_\tau(x) .
 \end{aligned} \quad (5.31)$$

Using (5.16) and the definition of ℓ_τ , for the last term we have

$$\begin{aligned}
 \frac{d}{d\tau|_0} D_w \ell_\tau(x) &= \frac{d}{d\tau|_0} D_w \left(\int_M \mathcal{L}(x, F_\tau(y)) f_\tau(y) d\rho(y) - \frac{\nu}{2} \right) \\
 &= D_w \int_M \nabla_{2,\mathfrak{v}} \mathcal{L}(x, y) d\rho(y) \\
 &= -\frac{1}{2} (D_w^+ - D_{-w}^+) \int_M d\rho(y) \nabla_{1,\mathfrak{v}}^+ \mathcal{L}(x, y) + (D_w b)(x) \frac{\nu}{2} + \chi_{\mathfrak{w},\mathfrak{v}}(x) \\
 &= -\frac{1}{2} (D_w^+ - D_{-w}^+) \nabla_{1,\mathfrak{v}}^+ \left(\ell(x) + \frac{\nu}{2} \right) + (D_w b)(x) \frac{\nu}{2} + \chi_{\mathfrak{w},\mathfrak{v}}(x) \\
 &\stackrel{(\star)}{=} -\frac{1}{2} (D_w^+ - D_{-w}^+) D_v^+ \ell(x) + \chi_{\mathfrak{w},\mathfrak{v}}(x)
 \end{aligned}$$

where in (\star) we have used $D_w b(x) \ell(x) = 0$ by (5.28) and since ρ is a minimizer. Thus (5.31) implies

$$\begin{aligned}
 0 &= \frac{1}{2} D_v^+ (D_w^+ - D_{-w}^+) \ell(x) - \frac{1}{2} (D_w^+ - D_{-w}^+) D_v^+ \ell(x) + \chi_{\mathfrak{w},\mathfrak{v}}(x) \\
 &= \frac{1}{2} (D_{[v,w]}^+ - D_{-[v,w]}^+) \ell(x) + \chi_{\mathfrak{w},\mathfrak{v}}(x) ,
 \end{aligned}$$

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giving (5.30).)

Since $[\mathfrak{w}, \mathfrak{v}] \in \mathfrak{J}^{\text{diff}}$, $D_{[w,v]}\ell(x)$ exists, and since ρ is a minimizer, this implies that

$$\frac{1}{2}(D_{[w,v]}^+\ell(x) - D_{-[w,v]}^+\ell(x)) = D_{[w,v]}\ell(x) = 0$$

for every $x \in M$. Thus (5.30) gives the result. \square

This concludes the first approach to show that the stochastic term vanishes pointwise. For the second approach, working with the push-forward (4.59), the following definition is necessary. It allows us to exchange the order of differentiation in the proof of Proposition 5.4.4.

Definition 5.4.3. *A transformation $\Phi \in C^\infty((-s_{\max}, s_{\max}) \times \mathcal{F} \rightarrow \mathcal{F})$ is **sufficiently differentiable** with respect to a family (5.1) if the partial derivatives*

$$\partial_\tau(\ell \circ F \circ \Phi), \quad \partial_s(\ell \circ F \circ \Phi) \quad \text{and} \quad \partial_\tau \partial_s(\ell \circ F \circ \Phi)$$

exist in $(-\delta, \delta) \times (-s_{\max}, s_{\max})$ and if $\partial_\tau \partial_s(\ell \circ F \circ \Phi)$ is continuous in s and τ at the point $(s, \tau) = (0, 0)$.

Proposition 5.4.4. *Let $\mathfrak{w} = (c, w) \in \mathfrak{J}^{\text{diff}}$. Let $\mathfrak{v} \in \mathfrak{J}$ be the generator of a family (5.1) of minimizers which satisfies*

$$\nabla_{\mathfrak{w}(\tau)} \ell_\tau(z) = 0 \quad \text{for all } z \in M_\tau, \quad (5.32)$$

where $\mathfrak{w}(\tau)$ is the push-forward (4.66) of \mathfrak{w} . Assume that the flow Φ of w is sufficiently differentiable with respect to the family (5.1). Then

$$\chi_{\mathfrak{w}, \mathfrak{v}}(x) = 0.$$

Proof. From $\mathfrak{w} \in \mathfrak{J}^{\text{diff}}$, it follows that $\tilde{\mathfrak{w}} := (c + (D_w \log f_\tau), w) \in \mathfrak{J}^{\text{diff}}$. Hence the proof of Lemma 4.3.2 applies and we have

$$\begin{aligned} 0 &= \frac{d}{d\tau|_0} \nabla_{\mathfrak{w}} \left(\int_M f_\tau(x) \mathcal{L}(F_\tau(x), F_\tau(y)) f_\tau(y) d\rho(y) - \frac{\nu}{2} f_\tau(x) \right) \\ &= \frac{d}{d\tau|_0} \frac{d}{ds|_0} \left(\int_M f_\tau(\Phi_s(x)) \mathcal{L}(F_\tau(\Phi_s(x)), F_\tau(y)) f_\tau(y) d\rho(y) - \frac{\nu}{2} f_\tau(\Phi_s(x)) \right) \\ &\quad + c(x) \left(\int_M \frac{d}{d\tau|_0} f_\tau(x) \mathcal{L}(F_\tau(x), F_\tau(y)) f_\tau(y) d\rho(y) - \frac{\nu}{2} b(x) \right) \end{aligned}$$

The last line vanishes by (5.19). Hence we are left with

$$\begin{aligned}
 0 &= \frac{d}{d\tau|_0} \frac{d}{ds|_0} \left(\int_M f_\tau(\Phi_s(x)) \mathcal{L}(F_\tau(\Phi_s(x)), F_\tau(y)) f_\tau(y) d\rho(y) - \frac{\nu}{2} f_\tau(\Phi_s(x)) \right) \\
 &= \frac{d}{d\tau|_0} \frac{d}{ds|_0} \left(f_\tau(\Phi_s(x)) \ell_\tau(F_\tau(\Phi_s(x))) + \frac{\nu}{2} f_\tau(\Phi_s(x)) - \frac{\nu}{2} f_\tau(\Phi_s(x)) \right) \\
 &= \frac{d}{d\tau|_0} \frac{d}{ds|_0} f_\tau(x) \ell_\tau(F_\tau(\Phi_s(x))) + \frac{d}{d\tau|_0} \frac{d}{ds|_0} f_\tau(\Phi_s(x)) \ell_\tau(F_\tau(x)) \\
 &= \frac{d}{d\tau|_0} \frac{d}{ds|_0} f_\tau(x) \ell_\tau(F_\tau(\Phi_s(x))) ,
 \end{aligned} \tag{5.33}$$

where in the first step we have used the definition of ℓ_τ and in the last step we have argued similarly as in (5.18) to conclude that

$$\frac{d}{d\tau|_0} \frac{d}{ds|_0} f_\tau(\Phi_s(x)) \ell_\tau(F_\tau(x)) = 0$$

(using that (5.1) consists of minimizers by assumption). The conditions in Definition 5.4.3 imply that the assumptions of Theorem 9.41 of [Rud76] are satisfied so that we can exchange the s - and τ -derivative in (5.33),

$$\begin{aligned}
 0 &= \frac{d}{ds|_0} \frac{d}{d\tau|_0} f_\tau(x) \ell_\tau(F_\tau(\Phi_s(x))) = \frac{1}{2} \left(\frac{d^+}{ds|_0} + \frac{d^-}{ds|_0} \right) \frac{d}{d\tau|_0} f_\tau(x) \ell_\tau(F_\tau(\Phi_s(x))) \\
 &= \frac{1}{2} \frac{d^+}{ds|_0} \frac{d^+}{d\tau|_0} f_\tau(x) \left(\ell_\tau(F_\tau(\Phi_s(x))) - \ell_\tau(F_\tau(\Phi_{-s}(x))) \right) = \chi_{\mathfrak{w}, \mathfrak{v}}(x) ,
 \end{aligned}$$

giving the result. \square

Remark 5.4.5. (INTERPRETATION OF PROPOSITIONS 5.4.2 AND 5.4.4) The above results show that the non-differentiable linearized field equations (5.12) indeed generalize the linearized field equations (4.68) in the sense that either of the assumptions of

- $\mathfrak{w} \in \mathfrak{J}$ being differentiable with respect to the family (5.1) generated by \mathfrak{v} and $[\mathfrak{w}, \mathfrak{v}] \in \mathfrak{J}^{\text{diff}}$ (Proposition 5.4.2), or
- the flow of the vector component of \mathfrak{w} being sufficiently differentiable and the push-forward $\mathfrak{w}(\tau)$ being a differentiable jet for every $\tau \in (-\delta, \delta)$ (Proposition 5.4.4)

imply that the stochastic term vanishes. The fact that it does not suffice to demand $\mathfrak{w} \in \mathfrak{J}^{\text{diff}}$ to obtain the linearized field equations from the non-differentiable linearized field equations arises because in order to derive (4.68) from (4.65) one needs to exchange the order of differentiation as in the proof of Proposition 5.4.4. This is implicit in Definition 4.3.3. Indeed, Definition 4.3.3 does imply that for every solution of the linearized field equations we have $\chi_{\mathfrak{w}, \mathfrak{v}}(x) = 0$ for all $x \in M$ if $\mathfrak{w} \in \mathfrak{J}^{\text{diff}}$.

We remark that Example 4.3.1 shows that the push-forward $\mathfrak{J}_\tau^{\text{diff}}$, defined as in (4.59), and Definition 5.4.1 do not agree in general. Thus the two approaches considered in this section are distinct. Since the push-forward $\mathfrak{w}(\tau)$ of a differentiable jet \mathfrak{w} is not in general

differentiable with respect to the member ρ_τ of the family (5.1), Definition 5.4.1 and Proposition 5.4.2 could be considered to be more adequate than the Assumption (5.32) in Proposition 5.4.4. \diamond

5.5 The Symplectic Form and Hamiltonian Time Evolution

In this section, we study the symplectic form introduced in Section 4.3.3, in the context of the non-differentiable linearized field equations. Thus for any compact $\Omega \in \Sigma(\mathcal{F})$, we consider the mapping

$$\sigma_\Omega(\mathbf{u}, \mathbf{v}) = \int_\Omega d\rho(x) \int_{M \setminus \Omega} d\rho(y) \sigma_{\mathbf{u}, \mathbf{v}}(x, y) \quad (5.34)$$

with

$$\begin{aligned} \sigma_{\mathbf{u}, \mathbf{v}}(x, y) &:= \frac{1}{4} \sum_{s, s' = \pm} \sigma_{\mathbf{u}, \mathbf{v}}^{s, s'}(x, y) \quad \text{and} \\ \sigma_{\mathbf{u}, \mathbf{v}}^{s, s'}(x, y) &= \nabla_{1, \mathbf{u}}^s \nabla_{2, \mathbf{v}}^{s'} \mathcal{L}(x, y) - \nabla_{1, \mathbf{v}}^{s'} \nabla_{2, \mathbf{u}}^s \mathcal{L}(x, y) \end{aligned} \quad (5.35)$$

as defined in (4.74) and (4.77). (Recall that the name “symplectic form” for this mapping is justified by Section 4.2, where we show that if \mathcal{L} and ℓ are smooth, (5.34) indeed is a symplectic form.)

In Chapter 4, we have found that, given assumptions to ensure the existence of certain terms, for all \mathbf{u}, \mathbf{v} which are differentiable jets and solutions of the linearized field equations, this symplectic form vanishes,

$$\sigma_\Omega(\mathbf{u}, \mathbf{v}) = 0, \quad (5.36)$$

(Theorem 4.3.5) and is bilinear (Proposition 4.3.6). As explained in Section 1.2 of the introduction and in Section 4.2.3, taking the limit indicated in Figure 1.2 and assuming suitable decay properties of jets, (5.36) gives rise to a surface layer integral which is conserved under the evolution of the linearized field equations, the *Hamiltonian time evolution*.

The next theorem specifies how the stochastic term (5.13) relates to the Hamiltonian time evolution.

Theorem 5.5.1. *Let \mathbf{w} and \mathbf{v} be solutions of the non-differentiable linearized field equations (5.12). Then for any compact $\Omega \in \Sigma(\mathcal{F})$, the symplectic form (5.34) satisfies*

$$\sigma_\Omega(\mathbf{w}, \mathbf{v}) = \int_\Omega d\rho(x) \tilde{\chi}_{\mathbf{w}, \mathbf{v}}(x) - \int_\Omega d\rho(x) \tilde{\nabla}_{[\mathbf{w}, \mathbf{v}]} \ell(x), \quad (5.37)$$

where

$$\tilde{\chi}_{\mathbf{w}, \mathbf{v}}(x) = \frac{1}{2} (\chi_{\mathbf{w}, \mathbf{v}}(x) - \chi_{\mathbf{w}, -\mathbf{v}}(x) - \chi_{\mathbf{v}, \mathbf{w}}(x) + \chi_{\mathbf{v}, -\mathbf{w}}(x)). \quad (5.38)$$

Note that for $\tilde{\nabla}_{[\mathbf{w}, \mathbf{v}]}$, we have used Notation (5.6). We interpret this result in Re-

mark 5.5.4. An alternative form of the right hand side of (5.37) is given in Proposition 5.5.3.

Proof. Anti-symmetrizing (5.12) in \mathfrak{v} and $-\mathfrak{v}$ yields

$$\tilde{\nabla}_{\mathfrak{w}} \int_M d\rho(y) (\tilde{\nabla}_{1,\mathfrak{v}} + \tilde{\nabla}_{2,\mathfrak{v}}) \mathcal{L}(x, y) - \tilde{\nabla}_{\mathfrak{w}} \tilde{\nabla}_{\mathfrak{v}} \frac{\nu}{2} = \frac{1}{2} (\chi_{\mathfrak{w},\mathfrak{v}}(x) - \chi_{\mathfrak{w},-\mathfrak{v}}(x)) \quad (5.39)$$

Define

$$\sigma_{\mathfrak{w},\mathfrak{v}}(x) := \int_M d\rho(y) (\tilde{\nabla}_{1,\mathfrak{w}} \tilde{\nabla}_{2,\mathfrak{v}} - \tilde{\nabla}_{1,\mathfrak{v}} \tilde{\nabla}_{2,\mathfrak{w}}) \mathcal{L}(x, y). \quad (5.40)$$

Anti-symmetrizing (5.39) in \mathfrak{w} and \mathfrak{v} gives

$$\tilde{\nabla}_{[\mathfrak{w},\mathfrak{v}]} (\ell(x) + \frac{\nu}{2}) + \sigma_{\mathfrak{w},\mathfrak{v}}(x) - \tilde{\nabla}_{[\mathfrak{w},\mathfrak{v}]} \frac{\nu}{2} = \tilde{\nabla}_{[\mathfrak{w},\mathfrak{v}]} \ell(x) + \sigma_{\mathfrak{w},\mathfrak{v}}(x) = \tilde{\chi}_{\mathfrak{w},\mathfrak{v}}(x), \quad (5.41)$$

where we have used Assumption (vi) to exchange differentiation and integration. We now integrate (5.41) over a compact $\Omega \subset \mathcal{F}$. Assumption (v) implies that the integrals exist. Hence we have

$$\int_{\Omega} d\rho(x) \sigma_{\mathfrak{w},\mathfrak{v}}(x) = \int_{\Omega} d\rho(x) \tilde{\chi}_{\mathfrak{w},\mathfrak{v}}(x) - \int_{\Omega} d\rho(x) \tilde{\nabla}_{[\mathfrak{w},\mathfrak{v}]} \ell(x). \quad (5.42)$$

Since the integrand of (5.40) is anti-symmetric with respect to x and y , the integration over $\Omega \times \Omega$ vanishes,

$$\int_{\Omega} d\rho(x) \int_{\Omega} d\rho(y) (\tilde{\nabla}_{1,\mathfrak{w}} \tilde{\nabla}_{2,\mathfrak{v}} - \tilde{\nabla}_{1,\mathfrak{v}} \tilde{\nabla}_{2,\mathfrak{w}}) \mathcal{L}(x, y) = 0. \quad (5.43)$$

From (5.35) we have

$$\begin{aligned} \sigma_{\mathfrak{w},\mathfrak{v}}(x, y) &= \frac{1}{4} (\nabla_{1,\mathfrak{w}}^+ + \nabla_{1,\mathfrak{w}}^-) (\nabla_{2,\mathfrak{v}}^+ + \nabla_{2,\mathfrak{v}}^-) \mathcal{L}(x, y) \\ &\quad - \frac{1}{4} (\nabla_{1,\mathfrak{v}}^+ + \nabla_{1,\mathfrak{v}}^-) (\nabla_{2,\mathfrak{w}}^+ + \nabla_{2,\mathfrak{w}}^-) \mathcal{L}(x, y) = (\tilde{\nabla}_{1,\mathfrak{w}} \tilde{\nabla}_{2,\mathfrak{v}} - \tilde{\nabla}_{1,\mathfrak{v}} \tilde{\nabla}_{2,\mathfrak{w}}) \mathcal{L}(x, y). \end{aligned}$$

Together with (5.43), this implies

$$\sigma_{\Omega}(\mathfrak{w}, \mathfrak{v}) = \int_{\Omega} d\rho(x) \sigma_{\mathfrak{w},\mathfrak{v}}(x).$$

Therefore, (5.42) gives (5.37). \square

Thus, in general, the evolution according to the non-differentiable linearized field equations does not conserve the surface layer integrals (1.9). However, they are conserved if Definition 5.3.2 holds, as shown by the next proposition. Note that since Definition 5.3.2 makes a statement about minimizers, in the next proposition we need to assume that the jets \mathfrak{w} and \mathfrak{v} are generators of families (5.1) of minimizers.

Proposition 5.5.2. *If symmetric derivatives vanish macroscopically and $\mathfrak{w}, \mathfrak{v}$ are generators of families of minimizers of the form (5.1) (and hence solutions of the non-differentiable linearized field equations), for every compact $\tilde{\Omega} \in \mathcal{M}$, we have*

$$\sigma_{\tilde{\Omega}}(\mathfrak{w}, \mathfrak{v}) = 0 .$$

Proof. We again use that for a family (5.1) of minimizers, Definition 5.3.2 implies (5.24). Proceeding as in (5.25) and (5.26), we obtain

$$\begin{aligned} \int_{\tilde{\Omega}} \tilde{\chi}_{\mathfrak{w}, \mathfrak{v}}(x) d\rho(x) &= -\frac{1}{4} \int_{\tilde{\Omega}} \left((D_{[v, w]}^+ - D_{[v, -w]}^+) - (D_{[-v, w]}^+ - D_{[-v, -w]}^+) \right. \\ &\quad \left. - (D_{[w, v]}^+ - D_{[w, -v]}^+) + (D_{[-w, v]}^+ - D_{[-w, -v]}^+) \right) \ell(x) d\rho(x) \\ &= -\frac{1}{2} \int_{\tilde{\Omega}} \left(D_{[v, w]}^+ - D_{-[v, w]}^+ - D_{[w, v]}^+ + D_{-[w, v]}^+ \right) \ell(x) d\rho(x) \\ &= -\int_{\tilde{\Omega}} \left(D_{[v, w]}^+ - D_{-[v, w]}^+ \right) \ell(x) d\rho(x) = -2 \int_{\tilde{\Omega}} \tilde{D}_{[v, w]} \ell(x) d\rho(x) \\ &\stackrel{(\star)}{=} -2 \int_{\tilde{\Omega}} \tilde{\nabla}_{[\mathfrak{v}, \mathfrak{w}]} \ell(x) d\rho(x) = 2 \int_{\tilde{\Omega}} \tilde{\nabla}_{[\mathfrak{w}, \mathfrak{v}]} \ell(x) d\rho(x) , \end{aligned}$$

where in (\star) we have used (5.8) and $\ell|_M = 0$. In the last step we have used the definition of $\nabla_{[\mathfrak{w}, \mathfrak{v}]}^+$, (5.7). Hence (5.37) implies

$$\sigma_{\tilde{\Omega}}(\mathfrak{w}, \mathfrak{v}) = \int_{\tilde{\Omega}} \tilde{\nabla}_{[\mathfrak{w}, \mathfrak{v}]} \ell(x) d\rho(x) .$$

This term vanishes by (5.23),

$$\int_{\tilde{\Omega}} \tilde{\nabla}_{[\mathfrak{w}, \mathfrak{v}]} \ell(x) d\rho(x) = \int_{\tilde{\Omega}} \tilde{D}_{[w, v]} \ell(x) d\rho(x) = 0 ,$$

where we have again used (5.8) and $\ell|_M \equiv 0$. \square

Proposition 5.5.3. *If $\mathfrak{v} = (b, v)$, $\mathfrak{w} = (c, w)$ and Ω are as in Theorem 5.5.1, we have*

$$\begin{aligned} \sigma_{\Omega}(\mathfrak{w}, \mathfrak{v}) &= \int_{\Omega} d\rho(x) \left(b(x) \tilde{\nabla}_{\mathfrak{w}} \ell(x) - c(x) \tilde{\nabla}_{\mathfrak{v}} \ell(x) \right) \\ &\quad + \int_{\Omega} d\rho(x) \int_M d\rho(y) \left(\tilde{D}_{1, w} \tilde{\nabla}_{2, \mathfrak{v}} \mathcal{L}(x, y) - \tilde{D}_{1, v} \tilde{\nabla}_{2, \mathfrak{w}} \mathcal{L}(x, y) \right) . \end{aligned} \tag{5.44}$$

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Proof. To show (5.44), expand $\chi_{\mathfrak{w},\mathfrak{v}}(x)$ similarly to (5.20) to obtain

$$\chi_{\mathfrak{w},\mathfrak{v}}(x) = \frac{1}{2} b(x) \frac{d^+}{ds|_0} \left(\ell(\Phi_s(x)) - \ell(\Phi_{-s}(x)) \right) \quad (5.45)$$

$$+ \frac{1}{2} \frac{d^+}{ds|_0} \frac{d^+}{d\tau|_0} \left(\ell(F_\tau(\Phi_s(x))) - \ell(F_\tau(\Phi_{-s}(x))) \right) \quad (5.46)$$

$$+ \frac{1}{2} \frac{d^+}{ds|_0} \frac{d^+}{d\tau|_0} \left(\ell_\tau(\Phi_s(x)) - \ell_\tau(\Phi_{-s}(x)) \right). \quad (5.47)$$

The second term (5.46) yields a contribution to (5.38) of

$$\tilde{D}_{[w,v]}\ell(x).$$

Using (5.8), we see that the $\tilde{D}_{[w,v]}$ -terms in (5.42) cancel. Furthermore, for $x \in M$ we have

$$(\tilde{D}_w b)\ell(x) = (\tilde{D}_v c)\ell(x) = 0.$$

Hence the only contributions to the right hand side of (5.42) come from (5.45) and (5.47). In (5.38), they add up to

$$b(x) \tilde{D}_w \ell(x) - c(x) \tilde{D}_v \ell(x) + \tilde{D}_w \int_M \tilde{\nabla}_{2,\mathfrak{v}} \mathcal{L}(x, y) d\rho(y) - \tilde{D}_v \int_M \tilde{\nabla}_{2,\mathfrak{w}} \mathcal{L}(x, y) d\rho(y).$$

We add $b(x) c(x) \ell(x) - c(x) b(x) \ell(x)$ to this equation. This gives

$$b(x) \tilde{\nabla}_{\mathfrak{w}} \ell(x) - c(x) \tilde{\nabla}_{\mathfrak{v}} \ell(x) + \tilde{D}_w \int_M \tilde{\nabla}_{2,\mathfrak{v}} \mathcal{L}(x, y) d\rho(y) - \tilde{D}_v \int_M \tilde{\nabla}_{2,\mathfrak{w}} \mathcal{L}(x, y) d\rho(y).$$

Integrating over Ω yields (5.44). □

Remark 5.5.4. (INTERPRETATION OF THEOREM 5.5.1) To interpret Theorem 5.5.1, suppose that space-time M carries a suitable notion of Cauchy surfaces $(\Sigma_t)_{t \in \mathbb{R}}$. Denote the restriction of \mathfrak{J} to Σ_t by \mathfrak{J}_{Σ_t} . We define an operator $\mathcal{U}_{t_2, t_1} : \mathfrak{J}_{\Sigma_{t_1}} \rightarrow \mathfrak{J}_{\Sigma_{t_2}}$ by $\mathcal{U}_{t_2, t_1} \mathfrak{u}_1 := \mathfrak{u}|_{\Sigma_{t_2}}$, where $\mathfrak{u} \in \mathfrak{J}^{\text{field}}$ is such that $\mathfrak{u}|_{\Sigma_{t_1}} = \mathfrak{u}_1$. Put differently, \mathcal{U}_{t_2, t_1} represents the time evolution, as specified by the non-differentiable linearized field equations (5.12). (This operator might not be unique.) As explained in the context of (1.9) in the introduction, we can introduce a symplectic form on Σ_t by choosing $\Omega = \Omega_t$ in (5.34), where Ω_t is the past of the Cauchy surface Σ_t . If suitable decay conditions hold for the solutions of the non-differentiable linearized field equations as one approaches infinity on the Cauchy surfaces (“spatial infinity”), this surface layer integral exists and Theorem 5.5.1 and Proposition 5.5.2 apply.

Theorem 5.5.1 implies that

$$\sigma_{\Sigma_{t_2}}(\mathcal{U}_{t_2, t_1} \mathfrak{u}_1, \mathcal{U}_{t_2, t_1} \mathfrak{v}_1) \neq \sigma_{\Sigma_{t_1}}(\mathfrak{u}_1, \mathfrak{v}_1)$$

for all t_1, t_2 and $\mathfrak{u}_1, \mathfrak{v}_1 \in \mathfrak{J}_{\Sigma_{t_1}}$ in general. I.e., the the time evolution operator \mathcal{U}_{t_2, t_1} does

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not constitute a symplectomorphism with respect to the symplectic form (5.34) in general. If however, Definition 5.3.2 holds, the situation is different whenever t_1 and t_2 are such that the time-strip between them is a macroscopic region $\tilde{\Omega} \in \mathcal{M}$ as in Definition (5.3.1). Then Proposition (5.5.2) implies that

$$\sigma_{\Sigma_{t_2}}(\mathcal{U}_{t_2,t_1} \mathbf{u}_1, \mathcal{U}_{t_2,t_1} \mathbf{v}_1) = \sigma_{\Sigma_{t_1}}(\mathbf{u}_1, \mathbf{v}_1)$$

for all $\mathbf{u}_1, \mathbf{v}_1 \in \mathfrak{J}_{\Sigma_{t_1}}$.

As mentioned in the context of Definition 5.3.1, the specification of what is ‘macroscopic’ depends on the application. (In fact, in principle it is possible to prove or disprove (5.21) by investigating properties of general minimizers of the causal action principle. Hence one might be able to determine from first principles which regions $\tilde{\Omega}$ are ‘macroscopic’ in the sense that (5.21) is satisfied.) Based on contemporary physics, it is reasonable to suppose that a necessary requirement for a region to be macroscopic is that its diameter is many orders of magnitude larger than the Planck length. In this case, Proposition 5.5.2 would imply that for times which are separated by an interval which is many orders of magnitude larger than the Planck time, the evolution is in fact a symplectomorphism, and differences from the Hamiltonian time evolution disappear.

To understand what this means, let us compare the time evolution operator \mathcal{U}_{t_2,t_1} to the time evolution operator $U(t_2, t_1)$ in non-relativistic quantum theory and the property that \mathcal{U}_{t_2,t_1} is a symplectomorphism to the property that $U(t_2, t_1)$ is unitary. Then the above explanations would compare to a situation where microscopically, the unitary evolution is broken by a stochastic term, opening the doors for “dynamical reduction” to occur (cf. Section 1.4). Macroscopically, however, the stochastic contributions cancel, giving rise to a unitary time evolution on time-scales which are many orders of magnitude larger than the Planck time. \diamond

5.6 Second Order Terms

In this section, we study higher order corrections to the non-differentiable linearized field equations. For the connection to quantum theory (see Section 1.4), it is sufficient to consider second order terms, and for simplicity and clarity we restrict the following analysis to this order.

A systematic expansion of the weak Euler-Lagrange equations in the differentiable case can be found in [Fin17]. Since the ansatz in [Fin17] differs from the following ansatz, we include some preliminary and motivating remarks. A short comparison of the two approaches can be found in Remark 5.6.1.

5.6.1 Preliminaries

Recall that integral curves of a vector field $v \in \Gamma(T\mathcal{F})$ are defined as maps $\gamma : \mathbb{R} \supset J \rightarrow \mathcal{F}$ such that $\gamma(t)' = v|_{\gamma(t)}$ for all $t \in J$, where J is an open interval. A *flow domain* on \mathcal{F} is an open subset $\mathcal{D} \subset \mathbb{R} \times \mathcal{F}$ such that for every $x \in \mathcal{F}$, the set $\mathcal{D}_x := \{t \in \mathbb{R} | (t, x) \in \mathcal{D}\}$ is

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an open interval containing 0 (cf. e.g. [Lee13, Chapter 12]). A *flow* on \mathcal{F} is a smooth map $F : \mathcal{D} \rightarrow \mathcal{F}$ that satisfies

$$\begin{aligned} F_0(x) &= x && \text{for all } x \in \mathcal{F} \\ F_{\tau'}(F_{\tau}(x)) &= F_{\tau'+\tau}(x) && \text{for all } \tau \in \mathcal{D}_x \text{ and } \tau' \in \mathcal{D}_{F_{\tau}(x)}, \end{aligned}$$

where as before we have used the notation $F_{\tau}(x) = F(\tau, x)$. A flow is maximal if it cannot be extended to a larger flow domain. It follows that the *infinitesimal generator* v of a flow F defined by

$$v|_x = (F|_x)'(0),$$

where $F|_x(\tau) = F(\tau, x)$ is the orbit of x under F , is a smooth vector field on \mathcal{F} . Crucial for the following ansatz is that the reverse is also true. Even so the infinitesimal generator is defined to be the derivative of the flow at $\tau = 0$, it uniquely determines the flow F in the following sense.

Theorem: ([Lee13, Theorem 12.9]) Let v be a smooth vector field on a smooth manifold \mathcal{F} . Then there is a unique maximal flow whose infinitesimal generator is v .

Remark 5.6.1. (COMPARISON TO [Fin17]) The crucial difference between this approach and the approach in [Fin17] is that in [Fin17], one assumes that a smooth vector field \tilde{v} in a neighborhood of M is given. Since an extension to \mathcal{F} remains arbitrary, \tilde{v} does not specify a unique flow on \mathcal{F} and further information is necessary to determine it. Here, on the other hand, we are working with a smooth vector field v on \mathcal{F} , the freedom in extending \tilde{v} to \mathcal{F} is implicit in the choice of family (5.1). Which approach is suitable depends on the application. \diamond

5.6.2 Second Order Terms

Let $h \in C^\infty(\mathcal{F}, \mathbb{R})$ and let $v \in \Gamma(\mathcal{F})$ be the infinitesimal generator of a flow F . The above definitions imply that

$$v(h)|_{F_{\bar{\tau}}(x)} = \frac{d}{d\tau} h(F_{\tau}(x))|_{\tau=\bar{\tau}}, \quad (5.48)$$

where we have used that $F|_x(\tau)$ is an integral curve of v by [Lee13, Lemma 12.7(d)]. Since h is smooth and F is smooth (by definition of a flow), it follows that the right hand side of (5.48) is smooth in τ as well. Thus, we can take a further τ -derivative. This gives

$$\begin{aligned} \frac{d^2}{d\tau^2} h(F_{\tau}(x))|_{\tau=\bar{\tau}} &= \frac{d}{d\bar{\tau}} \left(\frac{d}{d\tau} h(F_{\tau}(x))|_{\tau=\bar{\tau}} \right) \Big|_{\bar{\tau}=\bar{\tau}} \\ &= \frac{d}{d\bar{\tau}} \left(v(h)|_{F_{\bar{\tau}}(x)} \right) \Big|_{\bar{\tau}=\bar{\tau}} = v(v(h))|_{F_{\bar{\tau}}(x)}. \end{aligned} \quad (5.49)$$

Clearly, this object is not a vector field. To align (5.49) with the usual notation in this chapter, we write

$$\frac{d^2}{d\tau^2} h(F_{\tau}(x))|_{\tau=\bar{\tau}} = D_{1,v} D_{1,v} h(F_{\bar{\tau}}(x)),$$

where as usual the first derivative also acts on the argument of the second.

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Note that whereas a vector field $v \in \Gamma(T\mathcal{F})$ determines its flow $F_\tau(x)$ for all $\tau \in \mathcal{D}_x$, this is not the case for the scalar component of the transformation (5.1). The first derivative \dot{f}_0 does of course not allow to deduce the full function f_τ . Therefore, higher τ -derivatives of f_τ appear in what follows. In order to present the result in an elegant way, for a given family (5.1), we define the *directional quadratic semi-derivative* for jets as

$$\nabla_{\mathbf{v}, \mathbf{v}}^+ := \ddot{f}_0 + 2\dot{f}_0 D_v^+ + D_v^+ D_v^+, \quad (5.50)$$

and respectively $\nabla_{i, \mathbf{v}, \mathbf{v}}^+$ to indicate that the directional quadratic semi-derivative acts on the i th argument of a function.

Lemma 5.6.2. *For any family (5.1) of measures with generator $\mathbf{v} = (b, v)$, any $x \in \mathcal{F}$ and any flow Φ , we have*

$$\begin{aligned} & \frac{1}{2} \frac{d^+}{ds|_0} \frac{d^2}{d\tau^2}|_0 \int_{M_\tau} d\rho_\tau(y) f_\tau(x) \left(\mathcal{L}(F_\tau(\Phi_s(x)), y) - \mathcal{L}(F_\tau(\Phi_{-s}(x)), y) \right) \\ &= \tilde{D}_w \int_M d\rho(y) \left(\nabla_{1, \mathbf{v}, \mathbf{v}}^+ + \nabla_{2, \mathbf{v}, \mathbf{v}}^+ + 2 \nabla_{1, \mathbf{v}}^+ \nabla_{2, \mathbf{v}}^+ \right) \mathcal{L}(x, y) \\ &- \int_M d\rho(y) \left((D_w \ddot{f}_0)(x) + 2 (D_w b)(x) (D_{1, v}^+ + D_{2, v}^+ + b(y)) \right) \mathcal{L}(x, y), \end{aligned}$$

where w is the generator of Φ .

Proof. We compute the second τ -semi-derivative of the left hand side of (5.14). We have

$$\frac{d^2}{d\tau^2}|_0 \int_{M_\tau} d\rho_\tau(y) f_\tau(x) \mathcal{L}(F_\tau(\Phi_s(x)), y) = \int_M d\rho(y) \mathcal{O}(x, y)$$

where

$$\begin{aligned} \mathcal{O}(x, y) &= \left(\ddot{f}_0(x) + D_{1, v}^+ D_{1, v}^+ + D_{2, v}^+ D_{2, v}^+ + \ddot{f}_0(y) \right) \mathcal{L}(\Phi_s(x), y) \\ &+ 2 \left(b(x) D_{1, v}^+ + b(x) D_{2, v}^+ + b(x) b(y) + D_{1, v}^+ D_{2, v}^+ + D_{1, v}^+ b(y) + b(y) D_{2, v}^+ \right) \mathcal{L}(\Phi_s(x), y) \\ &= \left(\nabla_{1, \mathbf{v}, \mathbf{v}}^+ + \nabla_{2, \mathbf{v}, \mathbf{v}}^+ + 2 \nabla_{1, \mathbf{v}}^+ \nabla_{2, \mathbf{v}}^+ \right) \mathcal{L}(\Phi_s(x), y). \end{aligned}$$

Here, Assumption (v) guarantees existence and we have exchanged the derivatives with integration using Assumption (vi). Note that the derivatives which act on the first argument of \mathcal{L} are evaluated at the space-time-point $\Phi_s(x)$. Therefore, taking the s -semi-derivative at $s = 0$ gives

$$\frac{d^+}{ds|_0} \frac{d^2}{d\tau^2}|_0 \int_{M_\tau} d\rho_\tau(y) f_\tau(x) \mathcal{L}(F_\tau(\Phi_s(x)), y) = \int_M d\rho(y) \mathcal{O}'(x, y)$$

with

$$\begin{aligned} \mathcal{O}'(x, y) &= D_{1,w}^+ \left(\nabla_{1,\mathbf{v},\mathbf{v}}^+ + \nabla_{2,\mathbf{v},\mathbf{v}}^+ + 2 \nabla_{1,\mathbf{v}}^+ \nabla_{2,\mathbf{v}}^+ \right) \mathcal{L}(x, y) \\ &\quad - \left((D_w \ddot{f}_0)(x) + 2 (D_w b)(x) (D_{1,v}^+ + D_{2,v}^+ + b(y)) \right) \mathcal{L}(x, y). \end{aligned}$$

Here, the conditions on (5.10) in Assumption (v) guarantee existence. Anti-symmetrizing in w gives the result. \square

Lemma 5.6.3. *For a family (5.1) of minimizers, $c \in C^\infty(\mathcal{F}, \mathbb{R})$ and $x \in M$, we have*

$$c(x) \int_M d\rho(x) \left(\nabla_{1,\mathbf{v},\mathbf{v}}^+ + \nabla_{2,\mathbf{v},\mathbf{v}}^+ + 2 \nabla_{1,\mathbf{v}}^+ \nabla_{2,\mathbf{v}}^+ \right) \mathcal{L}(x, y) - c(x) \ddot{f}_0(x) \frac{\nu}{2} = 0.$$

Proof. We consider again the equation

$$\begin{aligned} \int_{M_\tau} d\rho_\tau(y) f_\tau(x) \mathcal{L}(F_\tau(x), y) \\ = f_\tau(x) \left(\ell_\tau(F_\tau(x)) + \frac{\nu}{2} \right). \end{aligned}$$

Taking the $\frac{d^2}{d\tau^2}|_0$ semi-derivative of the left hand side yields as above

$$\int_M d\rho(x) \left(\nabla_{1,\mathbf{v},\mathbf{v}}^+ + \nabla_{2,\mathbf{v},\mathbf{v}}^+ + 2 \nabla_{1,\mathbf{v}}^+ \nabla_{2,\mathbf{v}}^+ \right) \mathcal{L}(x, y).$$

Concerning the right hand side, we can argue as in (5.18) to conclude

$$\frac{d^2}{d\tau^2}|_0 f_\tau(x) \ell_\tau(F_\tau(x)) = 0$$

for all $x \in M$. Hence we are left with

$$\frac{d^2}{d\tau^2}|_0 f_\tau(x) \left(\ell_\tau(F_\tau(x)) + \frac{\nu}{2} \right) = \ddot{f}_0(x) \frac{\nu}{2}.$$

Multiplying by $c(x)$ gives the result. \square

Theorem 5.6.4. (Full non-differentiable field equations to second order) *For every family (5.1) of minimizers with generator \mathbf{v} , any $x \in M$ and any $\mathfrak{w} \in \mathfrak{J}$, we have*

$$\begin{aligned} &\tilde{\nabla}_{\mathfrak{w}} \int_M d\rho(y) (\nabla_{1,\mathbf{v}}^+ + \nabla_{2,\mathbf{v}}^+) \mathcal{L}(x, y) - \tilde{\nabla}_{\mathfrak{w}} \nabla_{\mathbf{v}}^+ \frac{\nu}{2} \\ &+ \tilde{\nabla}_{\mathfrak{w}} \int_M d\rho(y) \left(\nabla_{1,\mathbf{v},\mathbf{v}}^+ + \nabla_{2,\mathbf{v},\mathbf{v}}^+ + 2 \nabla_{1,\mathbf{v}}^+ \nabla_{2,\mathbf{v}}^+ \right) \mathcal{L}(x, y) - \tilde{\nabla}_{\mathfrak{w}} \nabla_{\mathbf{v},\mathbf{v}}^+ \frac{\nu}{2} \\ &= \chi_{\mathfrak{w},\mathbf{v}}(x) + \chi_{\mathfrak{w},\mathbf{v}}^{(2)}(x), \end{aligned}$$

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where $\chi_{\mathfrak{w},\mathfrak{v}}(x)$ is as in (5.13) and

$$\chi_{\mathfrak{w},\mathfrak{v}}^{(2)}(x) = \frac{1}{2} \frac{d^+}{ds|_0} \frac{d^{2+}}{d\tau^2|_0} f_\tau(x) \left(\ell_\tau(F_\tau(\Phi_s(x))) - \ell_\tau(F_\tau(\Phi_{-s}(x))) \right).$$

Proof. Consider (5.14). Terms which arise from first derivatives in τ are given by Proposition 5.2.1. The terms which arise from second derivatives in τ are obtained by adding Lemmas 5.6.2 and 5.6.3. Furthermore, using that ρ is a minimizer, for every $x \in M$, we have

$$\begin{aligned} & \int_M d\rho(y) \left((D_w \ddot{f}_0)(x) + 2(D_w b)(x) (D_{1,v}^+ + D_{2,v}^+ + b(y)) \right) \mathcal{L}(x, y) + c(x) \ddot{f}_0(x) \frac{\nu}{2} \\ &= (\nabla_{\mathfrak{w}} \ddot{f}_0)(x) \frac{\nu}{2} + 2(D_w b)(x) \left(D_{1,v}^+ \ell(x) + \int_M d\rho(y) \nabla_{2,v}^+ \mathcal{L}(x, y) \right) \\ &\stackrel{(\star)}{=} (\nabla_{\mathfrak{w}} \ddot{f}_0)(x) \frac{\nu}{2} + 2(D_w b)(x) \left(\nabla_{1,v}^+ \ell(x) + \int_M d\rho(y) \nabla_{2,v}^+ \mathcal{L}(x, y) \right) \end{aligned}$$

where in (\star) we have added the term $2(D_w b)(x) b(x) \ell(x)$, which vanishes for every $x \in M$. For the last term in brackets, we have

$$\nabla_{1,v}^+ \ell(x) + \int_M d\rho(y) \nabla_{2,v}^+ \mathcal{L}(x, y) = \int_M d\rho(y) (\nabla_{1,v}^+ + \nabla_{2,v}^+) \mathcal{L}(x, y) - \nabla_{\mathfrak{v}}^+ \frac{\nu}{2},$$

which vanishes by (5.19) since $\nabla_{\mathfrak{v}}^+ \frac{\nu}{2} = b(x) \frac{\nu}{2}$. Finally, using (5.50) the remaining term $(\nabla_{\mathfrak{w}} \ddot{f}_0)(x) \frac{\nu}{2}$ can be written as

$$\nabla_{\mathfrak{w}} \ddot{f}_0(x) \frac{\nu}{2} = \nabla_{\mathfrak{w}} \nabla_{\mathfrak{v},\mathfrak{v}}^+ \frac{\nu}{2} = \tilde{\nabla}_{\mathfrak{w}} \nabla_{\mathfrak{v},\mathfrak{v}}^+ \frac{\nu}{2},$$

giving the result. □

For an interpretation of Theorem 5.6.4 we refer to Section 1.4.

5.7 Noether-Like Theorems in the Non-Differentiable Setting

We conclude this chapter by extending Theorem 3.2.3 to our setting of jets which are not differentiable. Note that in Chapter 3, we have worked with a slightly different definition of ℓ compared to (4.45) (cf. (3.7)).

Theorem 5.7.1. *Let Φ_τ be a symmetry of the Lagrangian as in Definition 3.2.2. Then for any measure ρ and any compact $\Omega \in \Sigma(M)$, we have*

$$\frac{1}{2} \frac{d}{ds|_0} \int_\Omega d\rho(x) \int_{M \setminus \Omega} d\rho(y) \left(\mathcal{L}(\Phi_s(x), y) - \mathcal{L}(\Phi_{-s}(x), y) \right) = \int_\Omega d\rho(x) \tilde{D}_w \ell(x). \quad (5.51)$$

Proof. We proceed exactly as in the proof of Theorem 3.2.3. We multiply (3.16) by a bounded measurable function f on M with compact support and integrate. For all

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$s \in (-s_{\max}, s_{\max})$, this gives

$$\begin{aligned} 0 &= \iint_{M \times M} f(x) f(y) \left(\mathcal{L}(x, \Phi_s(y)) - \mathcal{L}(\Phi_{-s}(x), y) \right) d\rho(x) d\rho(y) \\ &= \iint_{M \times M} f(x) f(y) \left(\mathcal{L}(\Phi_s(x), y) - \mathcal{L}(\Phi_{-s}(x), y) \right) d\rho(x) d\rho(y), \end{aligned}$$

where in the last step we used the symmetry in the arguments of the Lagrangian (Assumption (i) on page 82). Existence of this expression is given by Assumption (v). We replace $f(y)$ by $1 - (1 - f(y))$, multiply out and use the definition of ℓ , (4.45). We thus obtain

$$\begin{aligned} 0 &= \int_M f(x) \left(\ell(\Phi_s(x)) + \frac{\nu}{2} - \ell(\Phi_{-s}(x)) - \frac{\nu}{2} \right) d\rho(x) \\ &\quad - \iint_{M \times M} f(x) (1 - f(y)) \left(\mathcal{L}(\Phi_s(x), y) - \mathcal{L}(\Phi_{-s}(x), y) \right) d\rho(x) d\rho(y). \end{aligned}$$

Choosing f as the characteristic function of Ω , we obtain the identity

$$\begin{aligned} &\int_{\Omega} d\rho(x) \int_{M \setminus \Omega} d\rho(y) \left(\mathcal{L}(\Phi_s(x), y) - \mathcal{L}(\Phi_{-s}(x), y) \right) \\ &= \int_{\Omega} \left(\ell(\Phi_s(x)) - \ell(\Phi_{-s}(x)) \right) d\rho(x). \end{aligned}$$

Since this equation holds for all $s \in (-s_{\max}, s_{\max})$, its s -derivative exists. However, using Assumption (vi), on the right hand side we can only exchange the semi-derivatives with integration. Doing so gives the result. \square

The previous theorem shows that in general, the non-differentiability of \mathcal{L} and ℓ leads to a break-down of the conservation law (3.17) and thus, by Sections 3.4 and 3.5, to a break-down of current conservation and conservation of energy-momentum. (Recall that we had to include the a differentiability assumption in Theorem 3.4.2.) However, the Euler-Lagrange equations imply that the right hand side of (5.51) has a varying sign (cf. (5.11)), making Definition 5.3.2 applicable. The next proposition shows that, given this assumption, the conservation laws hold macroscopically.

Proposition 5.7.2. *If symmetric derivatives vanish macroscopically, for every minimizer ρ of the causal variational principle and every compact macroscopic $\tilde{\Omega} \in \mathcal{M}$, we have*

$$\frac{d}{ds}|_0 \int_{\tilde{\Omega}} d\rho(x) \int_{M \setminus \tilde{\Omega}} d\rho(y) \left(\mathcal{L}(\Phi_s(x), y) - \mathcal{L}(\Phi_{-s}(x), y) \right) = 0.$$

Proof. Definition 5.3.2 implies

$$\int_{\tilde{\Omega}} d\rho(x) \tilde{D}_w \ell(x) = 0,$$

5 *Stochastic and Non-Linear Correction Terms*

where we have exchanged differentiation and integration with Assumption (vi). Hence Theorem 5.7.1 gives the result. \square

Chapter 6

Outlook

We conclude this thesis by giving a list of research projects which could be carried out based on the results which have been established in this thesis.

Connected to the Continuum Limit:

- Most importantly, it is necessary to evaluate the jet-formalism, the linearized field equations and the correction terms of Theorem 5.6.4 in the continuum limit. If successful, this will provide correction terms to fundamental equations of contemporary physics.
- Connected to this is that our formalism might be a suitable starting point for getting the connection to the canonical formulation of quantum field theory in Fock spaces. Working out physical applications in this context might ultimately contribute to the goal of making experimental predictions in the form of corrections to measurable physical quantities.

Internal to the Theory:

- One could derive higher order corrections to the linearized field equations. In particular, it seems promising to check whether higher order field equations, as well as higher order stochastic terms, can be controlled by Assumption 5.3.1 or similar assumptions, and how the higher order terms relate to the Hamiltonian time evolution. In case it would turn out that these higher order corrections cannot be suppressed by reasonable assumptions, anticipating an analysis in the continuum limit, this might hint towards correction terms to the fundamental equations of contemporary physics which cannot be “argued away”.
- In the same direction, it would be very interesting to study whether Definition 5.3.1 can be proven or disproven for minimizers of the causal action. Even if this turns out to be impossible at the time due to computational limitations, the investigation of minimizing examples might give hints for one or the other case. To see why research in this direction might be relevant, suppose that one were able to prove that Assumption 5.3.1 cannot hold for minimizers of the causal action. This would imply

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that the major argument for the negligibility of the stochastic term on macroscopic scales would break away, again paving the path to new experimental predictions. Put differently, investigations into the validity of Assumption 5.3.1 might yield lower bounds for the correction terms established in Chapter 5.

- ▶ It would be promising to establish a direct connection between Noether-like theorems and the dynamics of jets. More concretely, one might ask in which way the existence of symmetries of the Lagrangian or of symmetries of the universal measure is related to conservation laws for solutions of the non-differentiable linearized field equations.
- ▶ One could follow up on the example constructed in Section 4.4. E.g., one could modify the Lagrangian (4.78) to make it compatible with the assumptions of Chapter 5. This would allow to study the stochastic and non-linear correction terms in an example. Furthermore, one could compare this calculation to numerical analysis.

Concerning Foundations of Quantum Theory:

As explained in Section 1.4, the results of this thesis suggest a particular resolution of the quantum mechanical measurement problem based on the theory of causal fermion systems. Of course, the exact investigation of the form of the stochastic term in the continuum limit (cf. above) is one of the most promising research projects with respect to foundations of quantum theory, as it might give rise to a relativistic and interacting dynamical collapse model. Furthermore, the following questions seem promising to study from the perspective of causal fermion systems:

- ▶ Can the Born rule be derived from the mathematical structure of the theory of causal fermion systems? Or can it be shown to emerge when minimizing the causal action principle? Is there higher order interference?
- ▶ Is there any explanation, based on the theory of causal fermion systems, of why quantum theory determines the tensor product to be the right structure to compose systems?
- ▶ Which stochastic processes are compatible with the stochastic term derived in this thesis?

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